Crowns in bipartite graphs

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Abstract

A set $S \subseteq V(G)$ is stable (or independent) if no two vertices from $S$ are adjacent. Let $\Psi(G)$ be the family of all local maximum stable sets [6] of graph $G$, i.e., $S \in \Psi(G)$ if $S$ is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of $S$. If $I$ is stable and there is a matching from $N(I)$ into $I$, then $I$ is a crown of order $|I| + |N(I)|$, and we write $I \in Crown(G)$ [1].

In this paper we show that $\text{Crown}(G) \subseteq \Psi(G)$ holds for every graph, while $\text{Crown}(G) = \Psi(G)$ is true for bipartite and very well-covered graphs. For general graphs, it is $\text{NP}$-complete to decide if a graph has a crown of a given order [13]. We prove that in a bipartite graph $G$ with a unique perfect matching, there exist crowns of every possible even order.

Keywords: maximum matching, bipartite graph, König-Egerváry graph, crown, order of a crown, local maximum stable set.

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1 Introduction

Throughout this paper $G$ is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of $G$ induced by $X$. The set $N_G(v)$ is the neighborhood of $v \in V(G)$, while $N_G[v] = N_G(v) \cup \{v\}$. If $|N_G(v)| = 1$, then $v$ is a leaf, otherwise $v$ is internal. The neighborhood $N_G(A)$ is $\{v \in V(G) : N_G(v) \cap A \neq \emptyset\}$, and $N_G[A] = N_G(A) \cup A$. If $A, B \subset V(G)$, then $(A, B)$ denotes the set $\{ab : ab \in E(G), a \in A, b \in B\}$. A matching is a set of pairwise non-incident edges of $G$. The matching number $\mu(G)$ is the size of a maximum matching (a matching with the largest possible number of edges). A matching covering all the vertices is called perfect.

**Proposition 1.1** [6] Every tree contains a maximum matching covering all its internal vertices.

If $M$ is the unique perfect matching in the subgraph induced by the vertices that it saturates, then $M$ is a uniquely restricted matching [5]. A stable set of maximum size, denoted $\alpha(G)$, is a maximum stable set, and by $\Omega(G)$ we mean the family of all maximum stable sets. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then $G$ is a König-Egerváry graph. Each bipartite graph is König-Egerváry.

**Theorem 1.2** [10] $G$ is a König-Egerváry graph if and only if each maximum matching of $G$ is contained in $(S, V(G) - S)$, for every $S \in \Omega(G)$.

A set $A \subseteq V(G)$ is local maximum stable in $G$ if $A \in \Omega(G[N_G[A]])$ [6]. Let $\Psi(G)$ be the family of all local maximum stable sets of the graph $G$.

**Theorem 1.3** [11] Every $A \in \Psi(G)$ is a subset of some $S \in \Omega(G)$.

Recall that $G$ is a well-covered graph if all its maximal stable sets are of the same cardinality [12], and $G$ is very well-covered if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$ [4].

**Theorem 1.4** [8] If $G$ is a very well-covered graph, then $G[N_G[S]]$ is a König-Egerváry graph, for every $S \in \Psi(G)$.

If $S_j \in \Psi(G)$ for all $j \in \{1, \ldots, k\}$, and $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{k-1} \subset S_k$, then $\{S_j : 0 \leq j \leq k\}$ is called an accessibility chain for $S_k$.

**Theorem 1.5** [6] For trees, every $S \in \Psi(T)$ has an accessibility chain.

If $I$ is a stable set of $G$ such that there exists a matching from $N_G(I)$ into $I$, then $I$ is called a crown of $G$, and the number $|I| + |N_G(I)|$ is called the order of the crown $I$ [1]. Let $\text{Crown}(G) = \{I : I \subseteq V(G) \text{ such that } I \text{ is a crown}\}$. It is clear that $\emptyset, \{v\} \in \text{Crown}(G)$, where $v$ is an isolated vertex of $G$. 

Theorem 1.6 [13] Given a graph $G$ and an integer $k$, it is $NP$-complete to decide whether $G$ contains a crown whose order is exactly $k$.

In this paper we show that there exists a close relationship between crowns and local maximum stable sets of a graph. These two concepts are of importance in developing both fixed parameter and local optimal algorithms. It is known that the vertex cover problem is $NP$-complete, while in the context of parameterized complexity this problem is fixed parameter tractable [3]: the size of the vertex cover problem can be substantially reduced by deleting the vertices of $I \cup N(I)$ and their adjacent edges, where $I \in Crown(G)$. On the other hand, considering a local maximum stable set is a proper local optimization step for the $NP$-hard weighted maximum stable set problem [2]. In [6] we have shown that the family $\Psi(T)$ of a forest $T$ forms a greedoid on its vertex set. Bipartite and triangle-free graphs whose $\Psi(G)$ are greedoids, were treated in [7,9], respectively. Paying attention to Theorem 1.6, we prove that a bipartite graph $G$ with a unique perfect matching has crowns of all even possible number. The case of trees is analyzed as well.

2 Results

Lemma 2.1 $\text{Crown}(G) \subseteq \Psi(G)$ holds for every graph $G$.

Proof. Let $I \in \text{Crown}(G)$, i.e., $I$ is a crown. Then $I$ is a maximum stable in $G[N_G[I]]$. Therefore, $I \in \Psi(G)$, and finally, $\text{Crown}(G) \subseteq \Psi(G)$. \hfill $\square$

Lemma 2.1 allows to give an alternative proof for the following.

Theorem 2.2 [1] If $I$ is a crown of $G$, then there is a vertex cover of minimum size containing $N_G(I)$ and none of the vertices of $I$.

Proof. By Lemma 2.1, it follows that $I \in \Psi(G)$. Hence, by Theorem 1.3, there is some $S \in \Omega(G)$, such that $I \subseteq S$. Therefore, $N_G(I) \subseteq V(G) - S$, and this completes the proof, since $V(G) - S$ is a minimum vertex cover. \hfill $\square$

Fig. 1. Non-bipartite König-Egerváry graphs.

It is easy to see that $\text{Crown}(C_{2n}) = \Psi(C_{2n})$, while $\text{Crown}(C_{2n+1}) = \{\emptyset\} \neq \Psi(C_{2n+1})$. Let consider the graphs $G_1$ and $G_2$ from Figure 1. Notice that $\text{Crown}(G_1) \neq \Psi(G_1)$, because $\{a\} \in \Psi(G_1) - \text{Crown}(G_1)$, while $\text{Crown}(G_2) = \Psi(G_2)$. 


Theorem 2.3 Crown\((G) = \Psi(G)\) for bipartite and very well-covered graphs.

Proof. According to Lemma 2.1, it is enough to show that \(\Psi(G) \subseteq \text{Crown}(G)\).

Suppose that \(G\) is a bipartite graph. Let \(S \in \Psi(G)\), and \(S_A = S \cap A, S_B = S \cap B\), where \(\{A, B\}\) is the bipartition of \(G\). Since \(S\) is stable, there is no edge between vertices of \(S_A\) and \(S_B\). Now, we prove that there is a matching \(M_1\) from \(N_G(S_A)\) into \(S_A\). Assume, to the contrary, that there is no matching from \(N_G(S_A)\) into \(S_A\). By Hall’s Theorem, there must be some \(X \subseteq N_G(S_A)\), such that \(N_G(X) \cap S_A\) has \(|N_G(X) \cap S_A| < |X|\). Hence, the set \((S - N_G(X) \cap S_A) \cup X\) is a stable subset of \(N_G[S]\), larger than \(S\), which contradicts the fact that \(S \in \Psi(G)\). Therefore, a matching from \(N_G(S_A)\) into \(S_A\) must exist. In a similar way, one can show that there is a matching \(M_2\) from \(N_G(S_B)\) into \(S_B\). Further, we deduce that \(S \in \text{Crown}(G)\), because \(S\) is stable and \(M_1 \cup M_2\) is a matching from \(N_G(S)\) into \(S\).

Assume now that \(G\) is a very well-covered graph. If \(S \in \Psi(G)\), then \(S\) is stable and, by Theorem 1.4, \(G[N_G[S]]\) is a König-Egerváry graph. Hence, there is a matching from \(N_G(S)\) into \(S\). Consequently, \(S\) is a crown, i.e., \(\Psi(G) \subseteq \text{Crown}(G)\). \(\square\)

For every graph \(G\) with \(V(G) = \{v_i : 1 \leq i \leq n\}, n \geq 1\), and let \(H\) be the graph obtained by joining the vertex \(v_i\) to \(r_i\) copies of \(K_1\). Then \(H\) contains a crown of order \(k\), for every \(k \in \{0, 2, 3, ..., q\}\), where \(q = n + r_1 + r_2 + ... + r_n\). If \(T\) is a tree on \(n \geq 2\) vertices, then \(T\) contains crowns of order 2, defined by its leaves, and crowns of order \(n\), namely, every maximum stable set determines such a crown.

Proposition 2.4 [7] Let \(G\) be a bipartite graph and \(S_0 \in \Psi(G)\).

(i) Every maximum matching of \(G[N[S_0]]\) can be enlarged to a maximum matching in \(G\).

(ii) If \(G\) has a unique perfect matching, then each \(S \in \Psi(G)\) has an accessibility chain.

Proposition 2.5 If \(G\) is a bipartite graph with a perfect matching, then an order of a crown in \(G\) may be an even number only. Moreover, if \(G\) has a unique perfect matching, then it has crown of every possible even order.

Proof. Let \(I\) be a crown in \(G\) and \(M_0\) be a matching from \(N(I)\) into \(I\). Proposition 2.4(i) implies that \(M_0\) is contained in a maximum matching \(M\) of \(G\), because \(I \in \Psi(G)\), by Theorem 2.3. Since \(M\) is a perfect matching in \(G\), it follows that \(M_0\) is perfect in \(G[N[I]]\). Hence, \(|I| = |N(I)|\) and this ensures that the order of \(I\) is an even number.
Assume now that $M$ is the unique perfect matching of $G$, and let $S \in \Omega(G)$. By Proposition 2.4(ii), $S$ has an accessibility chain
\[ \emptyset \subset S_1 \subset S_2 \subset \ldots \subset S_{\alpha(G)-1} \subset S_{\alpha(G)} = S. \]

By Theorem 2.3, we have that every $S_k$ from this chain defines a crown. If $M_k$ is a maximum matching in $G[N[S_k]]$, then it can be enlarged to $M$, according to Proposition 2.4(i). Hence $M_k$ is a perfect matching in $G[N[S_k]]$, which implies that $S_k$ is a crown of order $2k$.

For bipartite graphs, having a perfect matching does not guarantee existence of crowns of each even order. For instance, $C_6$ has no crown of orders either two or four. For non-bipartite König-Egerváry graphs the situation is different. For example, the graph $G_1$ from Figure 2 has two perfect matchings and crowns of orders $0, 4, 6, 8$ only. On the other hand, the graph $G_2$ from Figure 2 has no perfect matching, but has crowns of order $k \in \{0, 2, 3, \ldots, 7\}$.

![G_1 and G_2](image)

Fig. 2. Both $G_1$ and $G_2$ are König-Egerváry graphs.

Clearly, each tree on $n \leq 2$ vertices has crowns of order $k \in \{0, n\}$.

**Theorem 2.6** If $T$ is a tree on $n \geq 3$ vertices, then $T$ contains a crown of order $k$, for every $k \in \{0, 2, 4, \ldots, 2\mu(T), 2\mu(T) + 1, 2\mu(T) + 2, \ldots, n\}$.

**Proof.** Let $S \in \Omega(T)$ be such that every leaf of $T$ is in $S$. By Proposition 1.1, there exists a maximum matching $M$ of $T$ covering all its internal vertices. Since $T$ is a tree, we have that $\mu(T) = |M| \leq |S| = \alpha(T)$, and $M$ matches all the vertices of $V(T) - S$ into $S$, by Theorem 1.2.

Theorem 1.5 ensures that there is an accessibility chain for $S$, say
\[ \emptyset = S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_{\mu(T)} \subset S_{\mu(T)+1} \subset \cdots \subset S_{\alpha(T)-1} \subset S_{\alpha(T)} = S. \]

By the choice of $M$, it follows that all the vertices in $S_{\alpha(T)} - S_{\mu(T)}$, if any, must be leaves. Consequently, $T$ has crowns of orders:

(a) $0, 2, 4, \ldots, 2\mu(T)$, defined respectively, by $S_k$, $k \in \{1, 2, \ldots, \mu(T)\}$, since $|S_k| = |N_T(S_k)|$;

(b) $2\mu(T) + 1, 2\mu(T) + 2, \ldots, n$, defined respectively, by $S_k$, $k \in \{2\mu(T) + 1, 2\mu(T) + 2, \ldots, \alpha(T)\}$. \qed
3 Concluding remarks

In this paper, some kinds of similarities between crowns and local maximum stable sets are under consideration. On the one hand, Theorem 2.3 claims that \( \text{Crown}(G) = \Psi(G) \) holds for both bipartite and very well-covered graphs. On the other hand, there exist non-bipartite and non-very well-covered graphs with \( \text{Crown}(G) = \Psi(G) \) (e.g., \( G_2 \) from Figure 1). It motivates the following.

**Problem 3.1** Characterize graphs satisfying \( \text{Crown}(G) = \Psi(G) \).

The path \( P_{2n} \) has crowns of order \( k \), for every \( k \in \{0, 2, 4, \ldots, 2n\} \), while \( P_{2n+1} \) contains crowns of order \( k \), for each \( k \in \{0, 2, 4, \ldots, 2n, 2n+1\} \). The crowns of \( K_{1,n-1} \) are of order \( k \), for every \( k \in \{0, 2, 3, \ldots, n\} \). It is clear that a graph has a crown of order 3 if and only if has two leaves with the same neighbor. This observation and Theorem 2.6 lead to the following.

**Conjecture 3.2** If \( T \) is a tree on \( n \geq 3 \) vertices, that has two leaves with the same neighbor, then \( T \) contains a crown of order \( k \), for every \( k \in \{0, 2, 3, \ldots, n\} \).

References


