Cerny conjecture for edge-colored digraphs with few junctions

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Abstract

In this paper we consider the Cerny conjecture in terminology of colored digraphs corresponding to finite automata. We define a class of colored digraphs having a relatively small number of junctions between paths determined by different colors, and prove that digraphs in this class satisfy the Cerny conjecture. We argue that this yields not only a new class of automata for which the Cerny conjecture is verified, but also that our approach may be viewed as a new more systematic way to attack the Cerny conjecture in its generality, giving an insight into the complexity of the problem.

We consider edge-colored digraphs on a set of vertices $V$ with the property that no two edges leaving a vertex have a common color. Such an assignment of colors to edges is called a road coloring. If for some color $\alpha$ occurring in the coloring there is no edge-colored $\alpha$ leaving a vertex $x$, then we assume, tacitly, that there is a loop at this vertex colored $\alpha$. Then, given a vertex $x$, each finite sequence of colors $\alpha_1, \ldots, \alpha_m$ (repetitions allowed) may be considered as a description of a path (road) starting in $x$ and reaching a (uniquely determined by this sequence) vertex $y \in V$.

We are interested in “universal instructions” making possible to reach a fixed vertex $y$ with no regard at which vertex we start. A sequence of colors $\alpha_1, \ldots, \alpha_m$ such that for each vertex $x$ it describes a path from $x$ to the given $y$ is called a synchronizing sequence (for the vertex $y$).

If such synchronizing sequences exist for every vertex $y$, then the digraph $G$ needs to be strongly connected and aperiodic. The celebrated Road coloring conjecture states that (for a digraph without multiple edges) these necessary conditions are also sufficient for existence of a coloring having a synchronizing sequence for every vertex $y \in V$. The conjecture has been proved recently by
The Cerny conjecture is stated in terms of finite automata [3] and is considered as the one of the most longstanding open problems in automata theory. So far, it has been proved for a few classes of finite automata, most of which have a graph-theoretic description. In particular, it has been proved for automata with eulerian underlying digraph [7], and for automata whose underlying digraph contains a monochromatic cycle consisting of all vertices [4]. In [6] we have shown that many other known results ([1,2,5,9,10]) can be unified as concerning automata preserving certain properties of intervals of an associated directed graph.

In this paper we define a new class of automata determined by a graph-theoretic property related to crossing paths of different colors, and prove that they satisfy the Cerny conjecture. We show that in spite of that the class is defined in a uniform way, it requires a few different types of algorithms to find a synchronizing sequence short enough. We believe that our result brings a new insight into the conjecture and may lead to a more systematic way to attack the Cerny conjecture in its generality.

1 Preliminaries

We exploit heavily the correspondence between automata, digraphs, and transformation monoids arising naturally in connection with the Cerny conjecture. Therefore, our terminology and notation is suitably mixed.

Let $A$ be a finite directed graph on a set of vertices $V$ (possibly with loops and multiple edges). By a road coloring of $A$ with a finite set of colors $\Sigma$ we mean an assignment of the colors to the edges, $\psi : E(A) \rightarrow \Sigma$, such that every two edges coming out from the same vertex have different colors. Tacitly, we assume that if no edge leaving a vertex $x$ has a color $\alpha \in \Sigma$, then there is a loop at $x$ colored $\alpha$. Under this assumption sequences of colors determine unique paths in the road colored digraph.

A color component of $A$ is a subgraph consisting of all edges of a given color after removing loops and isolated vertices. Such a component is identified with the color it represents. The set of vertices of a color component $\alpha$ is denoted $[\alpha]$.

For two different colors $\alpha$ and $\beta$, the vertices in $[\alpha] \cap [\beta]$ are called junctions. For a given color $\alpha$ a junction $x \in [\alpha]$ is called its out-junction, if there is an
edge of a different color leaving \( x \), and it is called an in-junction for \( \alpha \), if there is an edge of a different color entering \( x \). Of course, a junction may be both out- and in- for a given color, and there may be more colors intersecting in a junction. The degree of a junction is the number of different colors meeting in the junction minus one (that is, for a given color \( \alpha \), this is the number of other colors meeting \( \alpha \) at this junction). A road colored digraph \( A \) is called 2-junction, if for every color \( \alpha \) the sum of degrees of its junctions does not exceed 2. This means that each color has either two junctions of degree one, or one junction of degree at most two.

A road colored digraph \( A \) is called synchronizing if there exists a sequence of colors \( \alpha_1, \ldots, \alpha_m \), \((\alpha_i \in \Sigma)\), and a vertex \( y_0 \in A \) such that for every vertex \( x \in A \) the path starting in \( x \) and determined by the sequence \( \alpha_1, \ldots, \alpha_m \) leads to \( y_0 \). Such a sequence is called synchronizing. The Cerny conjecture, in terms of colored digraphs, states that for each synchronizing digraph \( A \) there exists a synchronizing sequence of length at most \((n-1)^2\), where \( n \) is the order of \( A \).

In this paper we prove the Cerny conjecture for 2-junction colored digraphs. This turns out to be a fairly complicated class containing some hard cases for which the Cerny conjecture has been already proved. In our proof we apply, in particular, the most advanced results on the Cerny conjecture obtained so far. In this extended abstract we show an example of the proof (proof of Theorem 2.2). The rest of the proofs can be found in a full version of the article.

We use mixed terminology speaking of colored digraphs or automata. For the latter, we also describe synchronization in terms of actions of words, which in many cases is simpler and more illuminating than using colored paths terminology.

Let \( A = \langle Q, \Sigma, \delta \rangle \) be a finite automaton on a set \( Q \), where \( Q \) is the set of states, \( \Sigma \) an alphabet, and \( \delta : Q \times \Sigma \to Q \) the transition function. The states are the vertices of the underlying digraph \( G \), whose edges are colored with the letters of \( \Sigma \). The fact that \( \delta(x, \alpha) = y \) corresponds to the edge \((x, y)\) colored \( \alpha \). It is clear that each automaton determines uniquely the underlying road colored digraph, and each road colored digraph determines uniquely an automaton \( A = \langle Q, \Sigma, \delta \rangle \).

In addition, we make use of a very useful correspondence with transition monoids. The letters/colors in \( \Sigma \) are identified (also) with transformations they induce on \( Q \). We write simply \( x\alpha \) to denote \( \delta(x, \alpha) \), which in terms of the underlying digraph is the end vertex of the edge starting in \( x \) and colored \( \alpha \).

A sequence of colors \( \alpha_1, \ldots, \alpha_m \) is identified with the word \( w = \alpha_1 \ldots \alpha_m \) over \( \Sigma \) and the corresponding path in the digraph. We write \( xw \) to denote the action of the word \( w \) on \( x \) (composed of actions of the successive letters).
2 Characterization

We start from a description of the structure of 2-junction digraphs, which helps us to distinguish suitable subclasses. We shall call a digraph \( A \) a cycle with a path if it is a union of a cycle \( C \) of length \( r \geq 0 \) and a path \( P \) of length \( p \geq 0 \) such that the \( C \cap P \) consists of the single vertex \( z \), which is the sink vertex of \( P \). We admit a trivial cycle \( r = 0 \) (then \( A \) is simply a path), and we admit a trivial path \( p = 0 \) (then \( A \) is simply a cycle). We do not admit \( r = 1 \), in which case the cycle would be a loop. First, let us observe the following.

**Lemma 2.1** If \( A \) is a strongly-connected 2-junction digraph, then every color component of \( A \) is a cycle with a path or a permutation consisting of at most two cycles.

Proof: Let \( \alpha \) be a color of \( A \). First of all note that, since \( A \) is strongly connected, if \( \alpha \) is not a cycle, than it has an out-junction on each its cycle and an in-junction at each its source. Now, if \( \alpha \) is a permutation then the claim is obvious. If it is a non-permutation, then it has at least one source, and at least one cycle (possibly reduced to one vertex, which must be different from the source, anyway). Therefore, it consists of exactly one cycle with one junction on it, and a path attached to this cycle with a junction at the beginning of this path.

Now we have

**Theorem 2.2** Let \( A \) be a synchronizing strongly-connected 2-junction digraph. Then the number of junctions in \( A \) is equal the number \( k \) of the colors of \( A \). Moreover, the junctions of \( A \) can be arranged in a cycle \( s_0, \ldots, s_{k-1} \) corresponding to an arrangement of colors \( \alpha_0, \ldots, \alpha_{k-1} \) of \( A \) in such a way that, modulo \( k \), for each \( i < k \), \( s_i \) is an in-coming junction of \( \alpha_i \) and the out-coming junction of \( \alpha_{i-1} \).

Proof: Since \( A \) is synchronizing, there is a color \( \alpha_0 \), which is not a cycle. By Lemma 2.1, \( \alpha_0 \) is a cycle with a nontrivial path, whose junctions are \( s_0 \) placed at the source, and \( s_1 \) placed on the cycle of \( \alpha_0 \). It follows that \( s_0 \neq s_1 \). Since \( A \) is strongly-connected there exists a directed path from \( s_1 \) to \( s_0 \).

If this path is just an edge \((s_1, s_0)\) belonging to a color \( \alpha_1 \), then both the colors \( \alpha_0 \) and \( \alpha_1 \) have two common junctions, and there is no room for these colors to have junctions with other colors (because \( A \) is two-junction). Since \( A \) is strongly connected, it follows that there are no other colors. Thus \( k = 2 \), and states \( s_0, s_1 \) with colors \( \alpha_0, \alpha_1 \) have the claimed properties.

Otherwise, let \( s_1, q_1^1, \ldots, q_{i_1}^1, \ldots, s_m, q_m^1, \ldots, q_{i_m}^m, s_0 \) be the shortest path from \( s_1 \) to \( s_0 \) with the notation chosen so that, for each \( j \), \( s_j, q_j^1, \ldots, q_{i_j}^j, s_{j+1} \) is a maximal sequence of consecutive vertices belonging to the same color. In par-
ticular, all \( s_j \) are junctions. It may happen that \( i_j = 0 \), and the path corresponding to a color consists of a single edge \((s_i, s_{i+1})\). Yet, since each color has at most two junctions, it follows that each color occurs on this path at most once. By the same reason, each of these colors is different from \( \alpha_0 \). Therefore adding the paths from \( s_0 \) to \( s_1 \) in \( \alpha_0 \) we obtain a cycle with junctions \( s_0, \ldots, s_m \). Since for each color occurring on the cycle there are two junctions of this color on the cycle, these colors have no other junctions, and therefore, similarly as before, there are no other colors in \( A \). Consequently, \( m = k - 1 \), and the result follows.

We note that if the color \( \alpha_i \) is a cycle with a path, then \( s_i \) is the junction in the source of the path, and \( s_{i+1} \) is the junction on the cycle of \( \alpha_i \). If \( \alpha_i \) is a cycle then \( s_i, s_{i+1} \) are two different states on the cycle (if they were the same state, the junction would be of degree more than 2).

3 Main result

The proof of the main result is divided into two parts. First, using Theorem 2.2 we obtain the following.

**Theorem 3.1** Let \( A = \langle Q, \Sigma, \delta \rangle \) be a synchronizing strongly-connected 2-junction automaton with \( n \) states. If the alphabet of \( A \) has more than 2 letters, then \( A \) has a synchronizing word of length not exceeding \((n - 1)^2\).

The proof of this result consists of three pages and will be presented in the full version of the article. Theorem 3.1 reduces our problem to automata with two-letter alphabets. Yet, the case of two letters turns out the most complicated one. We have the following:

**Theorem 3.2** Let \( A = \langle Q, \Sigma, \delta \rangle \) be a synchronizing strongly-connected 2-junction automaton with \( n \) states. If the alphabet of \( A \) has 2 letters, then \( A \) has a synchronizing word of length not exceeding \((n - 1)^2\).

The proof requires several lemmas and will be presented in the full version of the article.

References


