A Dirac-type theorem for Hamilton Berge cycles in random hypergraphs

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Abstract

A Hamilton Berge cycle of a hypergraph on n vertices is an alternating sequence $(v_1, e_1, v_2, \dots, v_n, e_n)$ of distinct vertices $v_1, \dots, v_n$ and distinct hyperedges $e_1, \dots, e_n$ such that $\{v_1, v_n\} \subseteq e_n$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for every $i \in [n-1]$. We prove a Dirac-type theorem for Hamilton Berge cycles in random r-uniform hypergraphs by showing that for every integer $r \geq 3$ there exists $k = k(r)$ such that for every $\gamma > 0$ and $p \geq \frac{\log^{k(r)}(n)}{n^{r-1}}$ asymptotically almost surely every spanning subhypergraph $H \subseteq H^{(r)}(n,p)$ with minimum vertex degree $\delta_1(H) \geq \left(\frac{1}{2^{r-1}} + \gamma\right) p\binom{n-1}{r-1}$ contains a Hamilton Berge cycle. The minimum degree condition is asymptotically tight and the bound on p is optimal up to possibly the logarithmic factor. As a corollary this gives a new upper bound on the threshold of $H^{(r)}(n,p)$ with respect to Berge Hamiltonicity.

Keywords: Random hypergraphs, Berge cycles, Dirac's theorem, resilience.

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1 Introduction

Many classical theorems of extremal graph theory give sufficient optimal minimum degree conditions for graphs to contain copies of large or even spanning structures. Lately it became popular to phrase such extremal results in terms of local resilience, where the *local resilience* of a graph G with respect to a given monotone increasing graph property $\mathcal{P}$ is defined as the minimum number $\rho \in \mathbb{R}$ such that one can obtain a graph without property $\mathcal{P}$ by deleting at most $\rho \cdot \deg(v)$ edges from every vertex $v \in V(G)$. For instance, using this terminology, Dirac's theorem [6] says that the local resilience of the complete graph K_n with respect to Hamiltonicity is $1/2 + o(1)$.

In recent years, an active and fruitful research direction in extremal and probabilistic combinatorics has become the study of resilience of random and pseudorandom structures. The systematic study of those with respect to various graph properties was initiated by Sudakov and Vu in [14], who in particular proved that $G(n,p)$ has resilience at least $1/2 - o(1)$ with respect to Hamiltonicity a.a.s. for $p > \log^4 n/n$. This result was improved by Lee and Sudakov [11] to $p \gg \log n/n$, which is essentially best possible with respect to both the constant $1/2$ and the edge probability, since one can find a.a.s. disconnected spanning subgraphs of $G(n,p)$ with degree at most $(1/2 - o(1))pn$ and since $G(n,p)$ itself is a.a.s. disconnected for $p \leq (1 - o(1)) \log n/n$.

A lot of resilience results are known for random graphs. For instance, the containment of triangle factors [3], almost spanning trees of bounded degree [2], pancyclic graphs [9], and almost spanning and spanning bounded degree graphs with sublinear bandwidth [1,5,8] were studied.

An *r-uniform hypergraph* is a tuple (V, E) with $E \subseteq \binom{V}{r}$ and thus the generalisation of a graph: the elements of V are called *vertices* and the elements of E *hyperedges* (or *edges* for short). It is therefore natural to ask for degree conditions that force a subhypergraph of the complete hypergraph to contain a copy of some given large structure. Such problems have been studied extensively in the last years, especially for different kinds of Hamilton cycles. Furthermore, (bounds on) the threshold for the existence of a Hamilton cycle in the random r-uniform hypergraph model $H^{(r)}(n,p)$ have been determined for various notions of cycles. We refer to [10] for an excellent survey by Kühn and Osthus of such problems.

To the best of our knowledge, there are no local resilience results for random hypergraphs at all so far. The purpose of this work is to provide a first such Dirac-type result in random hypergraphs.

We use $\deg(v)$ to denote the vertex degree of a vertex v in an r-uniform hypergraph H, i.e. for the number of hyperedges of H that contain v. The notion of resilience in graphs extends verbatim to the setting of hypergraphs.

We will be interested in resilience results of random r-uniform hypergraphs with respect to weak and Berge Hamiltonicity. A *weak cycle* is an alternating sequence $(v_1, e_1, v_2, \ldots, v_k, e_k)$ of distinct vertices $v_1, \ldots, v_k$ and hyperedges $e_1, \ldots, e_k$ such that $\{v_1, v_k\} \subseteq e_k$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for every $i \in [k-1]$. A weak cycle C is called *Berge cycle* if all its hyperedges are distinct. Moreover, we say that C is a *Hamilton Berge cycle* if $\{v_1, \ldots, v_k\}$ spans the whole vertex set of the hypergraph. A *weak Hamilton cycle* is defined in a similar way. There are other widely studied notions of cycles such as ℓ-overlapping cycles for $1 \leq \ell \leq r-1$, which we are not going to further mention here and instead refer to [10].

Surprisingly, the only result on the minimum vertex degree which implies the existence of a weak or a Berge Hamilton cycle is the one due to Bermond, Germa, Heydemann, and Sotteau [4]. They proved that for every integer $r \geq 3$ and $k \geq r+1$ any r-uniform hypergraph H with minimum vertex degree $\delta_1(H) \geq \binom{k-2}{r-1} + r - 1$ contains a Berge cycle on at least k vertices. If we ask for a Berge Hamilton cycle in an r-uniform hypergraph on n vertices, where r is fixed and n is large, then the bound $\binom{n-2}{r-1} + r - 1$ is very weak since it differs from the maximum possible degree by $\binom{n-2}{r-2} - r + 1$. Certainly, the two propositions below are folklore and should be known:

Proposition 1.1 *Let $r \geq 3$ and $n \geq r$ and let H be an r-uniform hypergraph on n vertices. If $\delta_1(H) > \binom{\lceil n/2 \rceil - 1}{r-1}$, then H contains a weak Hamilton cycle.*

The proof is a one-line argument by replacing every edge of H with a clique on r vertices and applying the original theorem by Dirac. The bound on the minimum vertex degree is sharp. Indeed, for even n, the disjoint union of two copies of the complete r-uniform hypergraph $K^{(r)}_{\frac{n}{2}}$ on $\frac{n}{2}$ vertices has minimum vertex degree $\binom{n/2-1}{r-1}$ but is disconnected. For odd n, the hypergraph H on n vertices that is the composition of two copies of $K^{(r)}_{\lceil \frac{n}{2} \rceil}$ that share one vertex satisfies $\delta_1(H) = \binom{\lceil n/2 \rceil - 1}{r-1}$ but does not contain a weak Hamilton cycle.

The following result can be obtained along the lines of the proof of Dirac's theorem for graphs.

Proposition 1.2 *Let $r \geq 3$ and let H be an r-uniform hypergraph on $n > 2r-2$ vertices. If $\delta_1(H) \geq \binom{\lceil n/2 \rceil - 1}{r-1} + n - 1$ then H contains a Hamilton Berge cycle.*

With a bit more effort, we can reduce the term $n-1$ in Proposition 1.2.

In any case, it follows from Propositions 1.1 and 1.2 that the resilience of the complete hypergraph $K_n^{(r)}$ is $1 - 2^{1-r} - o(1)$ with respect to both weak and Berge Hamiltonicity.

The threshold of the random r-uniform hypergraph $H^{(r)}(n,p)$ for weak Hamiltonicity has recently been shown by Poole [12] to be sharp and to be equal to $\frac{(r-1)! \log n}{n^{r-1}}$. Observe that below this threshold, the random hypergraph $H^{(r)}(n,p)$ has isolated vertices a.a.s., and is thus disconnected. Also this threshold is clearly a lower bound on the threshold for Berge Hamiltonicity.

Our main result is the following.

Theorem 1.3 *For each integer $r \geq 3$ and real $\gamma > 0$ there is a $k = k(r)$ such that the following holds a.a.s. for $\mathcal{H} = H^{(r)}(n,p)$ if $p \geq \frac{\log^k n}{n^{r-1}}$. Let $H \subseteq \mathcal{H}$ be a spanning subhypergraph with $\delta_1(H) \geq \left(\frac{1}{2^{r-1}} + \gamma\right) p \binom{n}{r-1}$. Then H contains a Hamilton Berge cycle. In particular H also contains a weak Hamilton cycle.*

By a similar consideration as in the dense case above, our minimum degree condition is asymptotically tight. Furthermore, the bound on the edge probability is optimal up to possibly some logarithmic factor. Moreover, it provides an alternative proof of the result in [12] with only slightly weaker edge probability.

The proof is based on the absorbing method developed by Rödl, Ruciński, and Szemerédi [13]. Of particular importance are the ideas from the proof of a Dirac-type result for random directed graphs due to Ferber, Nenadov, Noever, Peter and Škoric [7], which allow us to apply this method in such a very sparse scenario.

2 Outline of the proof of Theorem 1.3

First we need some more definitions. A *weak path* is an alternating sequence $(v_1, e_1, v_2, e_2, \ldots, v_k)$ of distinct vertices $v_1, \ldots, v_k$ and edges $e_1, \ldots, e_{k-1}$ such that $v_i, v_{i+1} \in e_i$ for every $i \in [k-1]$. A weak path is called *Berge path* if all its edges are distinct. Given a weak path $P = (v_1, e_1, \ldots, e_{k-1}, v_k)$, we denote by $V^*(P) := \{v_1, \ldots, v_k\}$ the set of vertices of the sequence of P.

We first explain the idea of the proof of Theorem 1.3 in the case of weak Hamilton cycles and sketch afterwards how we guarantee the cycle to be Berge.

Given a spanning subhypergraph $H \subseteq \mathcal{H}$ as in the statement of the theorem, we partition the vertex set of H into disjoint sets Y, Z and W, where Y and W are both of linear size and W contains most vertices of H. The set Z assumes the role of a reservoir and is of size $n/\log^{\mathcal{O}(1)} n$. Choosing a partition with such sizes uniformly at random guarantees that with high probability for every vertex v the edges incident to v are distributed as expected across and into the sets Y, Z and W. Also, since H is a subgraph of the random hypergraph, we have good control on the edge distribution among various subsets of vertices.

Next we construct a weak path Q with $V^*(Q) \subseteq Y$ such that for every subset $M \subseteq Z$ there exists a weak path Q_M that has the same endpoints as Q and such that $V^*(Q_M) = V^*(Q) \dot\cup M$. This property (*absorbing property*) will be crucial at a later stage of the argument.

Then we partition W randomly into $\log^{\mathcal{O}(1)} n$ sets and distribute $Y \setminus V^*(Q)$ among them such that all of these sets have the same size $n/\log^{\mathcal{O}(1)} n$. Informally speaking, since $|Y|$ is significantly smaller than $|W|$ and every vertex from Y is "well-connected" to W, such partition allows us to find weak paths $P_1, \ldots, P_m$, with $m = n/\log^{\mathcal{O}(1)} n$, so that $V^*(P_1), \ldots, V^*(P_m)$ form a partition of $W \dot\cup Y \setminus V^*(Q)$.

As a last step, we use vertices from Z to connect the paths $P_1, \ldots, P_m$ and Q into a weak cycle C (again this is possible since every vertex of H is "well-connected" into Z). Since the unused vertices M of Z can be absorbed by the path Q into a weak path Q_M with $V^*(Q_M) = V^*(Q) \dot\cup M$, we have found a weak Hamilton cycle in H in this way. To construct the path Q and to connect the paths $P_1, \ldots, P_m$ and Q into a cycle we will repeatedly use a lemma (connecting lemma) that will allow us to connect various vertices by paths of length $\mathcal{O}(\log n)$.

Now we describe the necessary changes to ensure that the cycle we actually construct is Berge. Recall that in this case, each edge is allowed to appear at most once in the cycle. Hence one needs to be careful every time a path is built, be it when the paths P_i are constructed simultaneously, when we apply the connecting lemma to connect the paths P_i and Q, and especially when we construct the paths Q and Q_M for $M \subseteq Z$. We resolve this problem by a careful analysis whenever we build paths simultaneously and by defining for each of these main steps a different bucket set $R \subseteq V(H)$ such that the edges of the paths constructed in that step only use vertices from the partite set, where the inner vertices of their sequences lie, plus vertices from R. By a careful choice of the bucket sets, it is then ensured that the Hamilton cycle that we get in the end is indeed Berge.

It is worth noting that it is not possible to merely reduce the problem of finding Berge cycles to a problem of finding a Hamilton cycle in the random graph to achieve the same resilience $1 - 2^{1-r} + o(1)$. The reason is that such a reduction leads to a resilience which is far from being the asymptotically optimal one, as asserted by Theorem 1.3.

References

[1] P. Allen, J. Böttcher, J. Ehrenmüller, and A. Taraz. The bandwidth theorem in sparse graphs. Manuscript in preparation.

[2] J. Balogh, B. Csaba, and W. Samotij. Local resilience of almost spanning trees in random graphs. *Random Structures & Algorithms*, 38(1-2):121–139, 2011.

[3] J. Balogh, C. Lee, and W. Samotij. Corrádi and Hajnal's theorem for sparse random graphs. *Combin. Probab. Comput.*, 21(1-2):23–55, 2012.

[4] J. C. Bermond, A. Germa, M. C. Heydemann, and D. Sotteau. Hypergraphes hamiltoniens. In *Colloques Internationaux CNRS, Univ. Orsay*, volume 260, pages 39–43. 1976.

[5] J. Böttcher, Y. Kohayakawa, and A. Taraz. Almost spanning subgraphs of random graphs after adversarial edge removal. *Combin. Probab. Comput.*, 22(05):639–683, 2013.

[6] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, 3(1):69–81, 1952.

[7] A. Ferber, R. Nenadov, A. Noever, U. Peter, and N. Škorić. Robust hamiltonicity of random directed graphs. *arXiv:1410.2198*, 2014.

[8] H. Huang, C. Lee, and B. Sudakov. Bandwidth theorem for random graphs. *J. Comb.Theory Ser. B*, 102(1):14–37, 2012.

[9] M. Krivelevich, C. Lee, and B. Sudakov. Resilient pancyclicity of random and pseudorandom graphs. *SIAM J. Discrete Math.*, 24(1):1–16, 2010.

[10] D. Kühn and D. Osthus. Hamilton cycles in graphs and hypergraphs: an extremal perspective. In *Proceedings of the International Congress of Mathematicians, Seoul, Korea*, volume 4, pages 381–406, 2014.

[11] C. Lee and B. Sudakov. Dirac's theorem for random graphs. *Random Structures & Algorithms*, 41(3):293–305, 2012.

[12] D. Poole. On weak hamiltonicity of a random hypergraph. *arXiv:1410.7446*, 2014.

[13] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):229–251, 2006.

[14] B. Sudakov and V. H. Vu. Local resilience of graphs. *Random Structures & Algorithms*, 33(4):409–433, 2008.