Tropical Catalan subdivisions

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Abstract

We revisit the associahedral subdivision of the Pitman-Stanley polytope to provide geometric realizations of the ν-Tamari lattice of Préville-Ratelle and Viennot (which generalizes the m-Tamari lattice) as the dual of a triangulation of a polytope, as the dual of a mixed subdivision and as the edge-graph of a polyhedral complex induced by a tropical hyperplane arrangement. The method generalizes to type B_n.

Keywords: m-Tamari lattice, associahedron, cyclohedron, triangulation $\Delta_n \times \Delta_m$

1 Introduction

The Tamari lattice, a partial order on Catalan sets, has received a lot of attention ever since it was first introduced in Dov Tamari's doctoral thesis in 1951 [14]. The covering relation corresponds to flips in polygon triangulations, rotations on binary trees and certain elementary transformation on Dyck paths.

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Many generalizations of the Tamari lattice have been proposed. The m-Tamari lattice, a recent generalization to Fuss-Catalan Dyck paths introduced by Bergeron and Préville-Ratelle [2], has raised a lot of interest. It has been further generalized by Préville-Ratelle and Viennot to the set of lattice paths above any given lattice path ν, giving rise to the ν-Tamari lattice [10].

One of the striking characteristics of the Tamari lattice is that its Hasse diagram can be realized as the skeleton of a polytope, the associahedron. Geometric realizations of some m-Tamari lattices were found by Bergeron "by hand" [1, Figures 4 and 6], who asked if similar constructions could be found for all the cases. Our main result is a positive answer to this question, providing three different approaches to realize ν-Tamari lattices.

Theorem 1.1 *Let ν be a lattice path from $(0,0)$ to (a,b). The ν-Tamari lattice $\mathrm{Tam}(\nu)$ can be realized geometrically as:*

(i) *the dual of a regular triangulation of a subpolytope of $\Delta_a \times \Delta_b$;*

(ii) *the dual of a coherent fine mixed subdivision of a generalized permutahedron (in $\mathbb{R}^a$ and in $\mathbb{R}^b$);*

(iii) *the edge graph of a polyhedral complex induced by an arrangement of tropical hyperplanes (in $\mathbb{TP}^a \cong \mathbb{R}^a$ and in $\mathbb{TP}^b \cong \mathbb{R}^b$).*

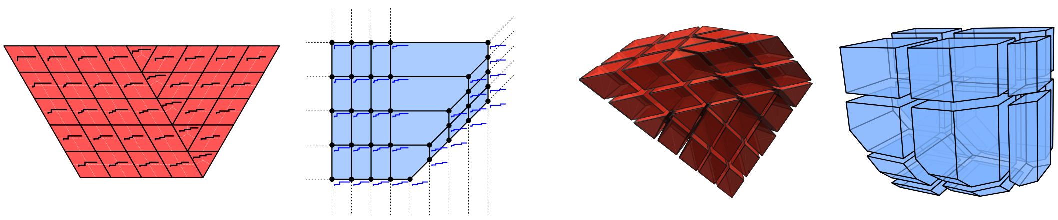

Figure 1. Realizations (ii) and (iii) of the 4-Tamari lattice for $n = 3$ (left) and the 2-Tamari lattice for $n = 4$ (right). They should be compared with [1, Fig. 4 and 6].

The tropical subdivision from (iii) should be considered as the ν-associahedron, the analogue of the associahedron for $\mathrm{Tam}(\nu)$. When $\nu = (\mathsf{NE})^n$, it is indeed an ordinary associahedron that is also tropically convex [5].

Our starting point is a ubiquitous triangulation $\mathfrak{A}_n$ of $\mathcal{C}_n$, the "upper triangular" subpolytope of the Cartesian product of two simplices:

$$\mathcal{C}_n = \mathrm{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) \colon 0 \le i \le \overline{j} \le n\right\} \subseteq \Delta_n \times \Delta_{\overline{n}} = \mathrm{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) \colon 0 \le i, \overline{j} \le n\right\}.$$

(Here, and throughout, $\mathbf{e}_i$ and $\mathbf{e}_{\overline{j}}$ denote standard basis vectors of $\mathbb{R}^{n+1}$ and we use over-lined variables to distinguish the indexes of the two factors.)

It is well known that $\mathfrak{A}_n$ is a flag regular triangulation whose dual simplicial complex is the join of a simplex with the $(n-1)$-dimensional (dual) associahedron. This is why we call it the *Associahedral triangulation*. It has been rediscovered several times, for diverse interpretations and objects related to $\mathcal{C}_n$. To the best of our knowledge, its first appearance was in [6] as a triangulation of a root polytope. Some of its later incarnations are as a fine mixed subdivision of the Pitman-Stanley polytope [13], as a triangulation of certain Gelfand-Tseltsin polytope [8] and of certain order polytope [11].

The approach of considering $\mathfrak{A}_n$ embedded in the product of two simplices has several advantages. One can consider its restriction to faces of $\Delta_n \times \Delta_{\overline{n}}$, which are also products of simplices. As we will see, for each lattice path ν from $(0,0)$ to (a,b) there is a pair $I, \overline{J} \subseteq [a+b], \overline{[a+b]}$ such that the restriction of $\mathfrak{A}_{a+b}$ to its face $\Delta_I \times \Delta_{\overline{J}}$ induces a triangulation dual to $\mathrm{Tam}(\nu)$. Its geometry gives alternative explanations for many properties of $\mathrm{Tam}(\nu)$ observed in [10].

As most "non-crossing objects", $\mathfrak{A}_n$ has a "non-nesting" analogue: the staircase triangulation of $\Delta_n \times \Delta_{\overline{n}}$ restricted to $\mathcal{C}_n$. It already appeared as the standard triangulation of an order polytope in [12], and was also considered in the references mentioned above. In our previous work [3] we applied a cyclic shift to this triangulation to produce the *Dyck path triangulation* of $\Delta_n \times \Delta_{\overline{n}}$ in our study of extendability of partial triangulations of $\Delta_n \times \Delta_{\overline{m}}$.

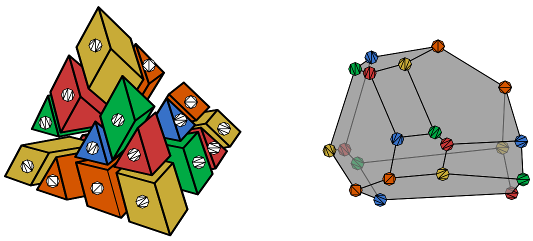

Figure 2. $\mathfrak{C}_4$ as a mixed subdivision of $4\Delta_3$ and as a tropical polytope.

If we apply the same cyclic procedure to $\mathfrak{A}_n$, we obtain a flag regular triangulation $\mathfrak{C}_n$ of $\Delta_n \times \Delta_{\overline{n}}$. Its maximal cells are indexed by centrally symmetric triangulations of a $(2n+2)$-gon and its dual complex is a cyclohedron. We call it the *Cyclohedral triangulation*. Restricting to its faces, we obtain type B_n analogues of the realizations of $\mathrm{Tam}(\nu)$.

2 The $(I, \overline{J})$-Associahedral triangulation

Recall that, under the standard bijection between vertices of $\Delta_n \times \Delta_{\overline{m}}$ and edges of the complete bipartite graph $K_{[n],[\overline{m}]}$ (where $[n] = \{0, 1, \ldots, n\}$ and

$[\overline{m}] = \{\overline{0}, \overline{1}, \dots, \overline{m}\}$), maximal simplices of $\Delta_n \times \Delta_{\overline{m}}$ correspond to spanning trees of $K_{[n],[\overline{m}]}$ (see [4, Sec. 6.2]).

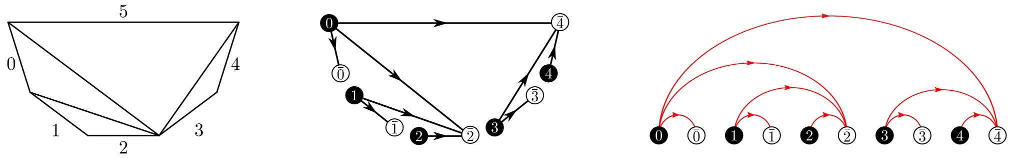

Figure 3. From a triangulation to a non-crossing alternating tree.

Figure 3 illustrates a procedure to assign a tree on $K_{[n],[\overline{n}]}$ to each triangulation of a convex $(n+2)$-gon. The collection of trees obtained this way index the cells of the *Associahedral triangulation* $\mathfrak{A}_n$. Its restriction to the face of $\Delta_n \times \Delta_{\overline{n}}$ indexed by $I \subseteq [n]$ and $\overline{J} \subset [\overline{n}]$ is a triangulation of

$$\mathcal{C}_{I,\overline{J}} := \operatorname{conv}\left\{(\mathbf{e}_i, \mathbf{e}_{\overline{j}}) \;:\; i \in I, \overline{j} \in \overline{J} \text{ and } i \leq j\right\}.$$

An $(I, \overline{J})$-*tree* is a maximal non-crossing alternating graph with support $I \sqcup \overline{J}$. That is, a maximal graph whose edges are of the form $(i, \overline{j})$ for $i \in I$ and $\overline{j} \in \overline{J}$ with $i \leq j$; and non-crossing (avoid $(i, \overline{j})$ and $(i', \overline{j}')$ if $i < i' \leq \overline{j} < \overline{j}'$).

Lemma 2.1 *The set of $(I, \overline{J})$-trees indexs a regular triangulation $\mathfrak{A}_{I,\overline{J}}$ of $\mathcal{C}_{I,\overline{J}}$.*

Consider now a partition of $[\ell]$ into two disjoint subsets I and $\overline{J}$ with $0 \in I$ and $\overline{\ell} \in \overline{J}$. We call such a partition a *canopy*. We define a lattice path $\nu(I, \overline{J})$ from $(0,0)$ to $(|I|-1, |\overline{J}|-1)$ as the path whose kth step is east if $k \in I$ and north if $\overline{k} \in \overline{J}$ for $1 \leq k \leq \ell - 1$. For example, $\nu(\{0,1,2,5,6,9\}, \{\overline{3},\overline{4},\overline{7},\overline{8},\overline{10}\}) = \mathsf{EENNEENNE}$.

We also associate to each $(I, \overline{J})$-tree T a lattice path $\rho(T)$ from $(0,0)$ to $(|I|-1, |\overline{J}|-1)$ as follows: For each $\overline{j} \in \overline{J}$, do $d_T(\overline{j}) - 1$ east steps and one north step, where $d_T(\overline{j})$ is the degree of $\overline{j}$ in T. Then remove the last north step. That is, if $\overline{J} = \{\overline{j}_1, \dots, \overline{j}_k\}$, then $\rho(T) = \underbrace{\mathsf{E} \cdots \mathsf{E}}_{d_T(\overline{j}_1)-1} \mathsf{N} \underbrace{\mathsf{E} \cdots \mathsf{E}}_{d_T(\overline{j}_2)-1} \mathsf{N} \cdots \underbrace{\mathsf{E} \cdots \mathsf{E}}_{d_T(\overline{j}_k)-1}$.

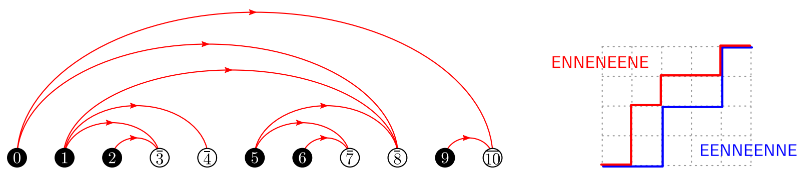

Figure 4. An $(I, \overline{J})$-tree T for $I = \{0,1,2,5,6,9\}$ and $\overline{J} = \{\overline{3},\overline{4},\overline{7},\overline{8},\overline{10}\}$). It corresponds to the $\nu(I, \overline{J})$-path $\rho(T) = \mathsf{ENNENEENE}$.

Proposition 2.2 *If $\nu = \nu(I, \overline{J})$ for a canopy $I, \overline{J}$, then ρ is a bijection from the set of $(I, \overline{J})$-trees to the set of ν-paths; and two $(I, \overline{J})$-trees are related by a flip if and only if the corresponding ν-paths are related by a ν-Tamari transformation (cf. [10]). Moreover, for each path ν there is a canopy $I, \overline{J}$ such that $\nu(I, \overline{J}) = \nu$.*

Notice that the combination of Lemma 2.1 with Proposition 2.2 already gives a realization of the ν-Tamari lattice as the dual of a $\mathfrak{A}_{I,\overline{J}}$. Via the Cayley trick [7], $\mathcal{C}_{I,\overline{J}}$ can be interpreted as a Minkowski sum of faces of a simplex (a generalized permutahedron [9]), and $\mathfrak{A}_{I,\overline{J}}$ becomes a mixed subdivivion. The duality between tropical hyperplane arrangements and regular triangulations of $\Delta_n \times \Delta_{\overline{m}}$ [5] gives our last realization, which explains the pictures in [1].

3 The Cyclohedral triangulation

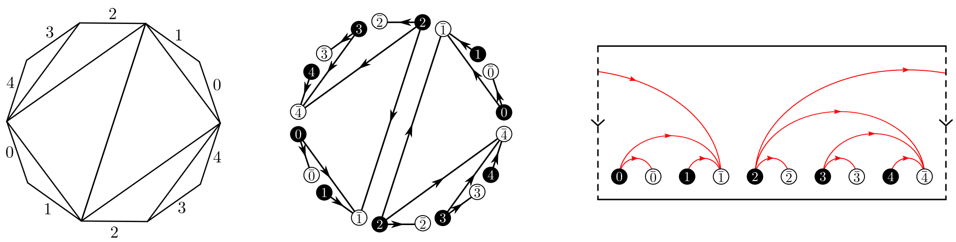

Figure 5. From a cs-triangulation to a non-crossing alternating tree on the cylinder.

Figure 5 shows how to obtain a tree on $[n] \cup [\overline{n}]$ from a centrally symmetric triangulation of a $(2n + 2)$-gon. If we collect all such trees as we did in $\mathfrak{A}_n$, we obtain a regular triangulation of $\Delta_n \times \Delta_{\overline{n}}$ dual to a cyclohedron: the *Cyclohedral triangulation* $\mathfrak{C}_n$. To restrict to faces of $\Delta_n \times \Delta_{\overline{m}}$, we cyclically order $I \cup \overline{J}$ and define cyclic $(I, \overline{J})$-trees to be alternating trees that are non-crossing when drawn on a cylinder, as the one in the picture. This provides a natural definition of $(I, \overline{J})$-cyclohedra.

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