Rainbow perfect matchings in r-partite graph structures

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1 Introduction

A latin transversal in a square matrix of order n is a set of entries, no two in the same row or column, which are pairwise distinct. A longstanding conjecture of Ryser states that every Latin square with odd order has a latin transversal. Some results on the existence of a large partial latin transversal can be found in [11,6,16]. Mainly motivated by Ryser's conjecture, Erdős and Spencer [8] proved the following result:

Theorem 1.1 *[8] Let A be a square matrix of order n in which no entry appears more than $\frac{n-1}{4e}$ times. Then A has a latin transversal.*

A matching M in an edge–colored graph is *rainbow* if no two edges in M have the same color. Theorem 1.1 can be rephrased by saying that every edge–coloring of the complete graph $K_{n,n}$

in which no color appears more than $(n-1)/4e$ times contains a rainbow perfect matching.

There has been a considerable amount of literature related to the study of existence and enumeration of rainbow matchings in graphs and hypergraphs. Bissacot et al. [5] improved the constant $\frac{1}{4e}$ in Theorem 1.1 to $\frac{27}{256}$ and their result was made algorithmic in [10]. Perarnau and Serra [14] provided bounds on the number of latin transversals under the hypothesis of Theorem 1.1. Here we are interested in analogous versions of Theorem 1.1 for other graph structures such as complete bipartite multigraphs, regular bipartite graphs and hypergraphs.

1.1 Complete multigraphs:

Aharoni and Berger [2] considered the problem of finding a rainbow matching from a slightly different perspective by asking how large a given collection $\mathcal{M}$ of matchings (not necessarily edge–disjoint) should be to ensure that a matching of some given size t can be obtained by using an edge of each of the matchings in $\mathcal{M}$. If every matching in $\mathcal{M}$ is given a different color, then we aim to find a rainbow matching of size t. The question, posed in the general context of r–partite r–uniform hypergraphs, was connected by Alon [4] to zero–sum problems in additive combinatorics by disproving a conjecture in [2] and providing some bounds on the size of $\mathcal{M}$ in terms of t. Glebov, Sudakov and Szabó [9] improved the upper bound in [4] from superexponential to polynomial in t. In the case of bipartite graphs ($r = 2$), Clemens and Ehrenmüller showed that $|\mathcal{M}| \geq 3t/2 + o(t)$ suffices [7].

Nevertheless, the main obstruction for a family $\mathcal{M}$ of matchings to contain a rainbow matching seems to be the existence of edges appearing in a large number of the matchings. This motivates the study of the existence of rainbow matchings in multigraphs with bounded multiplicity of edges. A first step in this direction in the context of Theorem 1.1 is the following result.

Theorem 1.2 *For every $n \geq 1$ and $m \geq 1$, let G be an edge–colored complete bipartite mutligraph with both stable sets of cardinality n and where each edge has multiplicity m. If each color appears less than $\frac{m(n-1)}{4e}$ times, then G has a rainbow perfect matching.*

1.2 Regular bipartite graphs:

Another direction of research considers general graphs other than the complete bipartite graph. The minimum colour degree is the smallest number of distinct colours on the edges incident with a vertex. Kostochka and Yancey [12] showed that every edge-coloured graph on n vertices with minimum colour degree at least k contains a rainbow matching of size at least k, provided $n \geq (17/4)k^2$, and this bound on n was reduced to $n \geq 4k-4$ by Lo and Tan [13].

Hall's theorem ensures that d–regular bipartite graphs have a perfect matching. Thus, given an edge–colored d–regular bipartite graph, one can aim at finding a perfect rainbow matching. An answer to that question under the setting of Theorem 1.1 and for relatively large d is given by following result.

Theorem 1.3 *For every $n \geq 1$ and $d \geq n/2+2$, let G be an edge–colored d-regular bipartite graph with both stable sets of cardinality n. If each color appears less than $\frac{(2d-n-3)^2}{4ed}$ times, then G has a rainbow perfect matching.*

1.3 r–partite r–uniform hypergraphs:

A rainbow matching in a bipartite graph can be understood as a matching in a certain 3–partite 3–uniform hypergraph by adding to each edge a third vertex identifying its color. The existence of matchings in r–partite r–uniform hypergraphs has also been intensively studied. A conjecture of Ryser relating the matching number of such an r–uniform hypergraph with the covering number was solved by Aharoni [1] for $r = 3$ by using a hypergraph version of Hall's theorem obtained by Aharoni and Haxel [3]. However, direct application of this result only guarantees the existence of a relatively small rainbow partial matching in the corresponding hypergraph. Our last result concerns rainbow matchings in complete r–partite r–uniform hypergraphs.

Theorem 1.4 *For every $n \geq 1$ and $r \geq 2$, let $K^{(r)}_{n,\ldots,n}$ be an edge–colored r–partite r–uniform hypergraph where each stable set has cardinality n. If each color appears less than $\frac{(n-1)^{r-1}}{2re}$ times, then $K^{(r)}_{n,\ldots,n}$ has a rainbow perfect matching.*

2 Lopsided Local Lovász Lemma

The three above theorems are proved by using a Lopsided version of the Local Lovász Lemma. This version was introduced by Erdős and Spencer [8] precisely to prove Theorem 1.1 and has become a powerful tool in probabilistic combinatorics. We use the following version of the result.

Let $\mathcal{E}$ be a family of events in a probability space and $p \in (0, 1)$. A graph H which has the events as vertices is a *p–lopsided* graph for $\mathcal{E}$ if, for each event $A \in \mathcal{E}$, and every subset $\mathcal{S} \subseteq \mathcal{E} \setminus N_H(A)$,

$$\mathbb{P}[A| \cap_{B\in\mathcal{S}} \overline{B}] \leq p.$$

Lemma 2.1 (LLLL) *Let $\mathcal{E}$ be a family of events in a probability space and let H be a p–lopsided graph for $\mathcal{E}$. Let d be the maximum degree of H. If*

$$ep(d + 1) \leq 1,$$

then

$$\mathbb{P}\left[\cap_{A\in\mathcal{E}}\overline{A}\right] > 0.$$

The proofs of Theorems 1.2, 1.3 and 1.4 follow the same strategy. We consider a matching M chosen uniformly at random and a set of events $\mathcal{E}$ consisting of pairs of edges in M with the same color. Therefore, if none of these events occur, M is a rainbow matching. The conclusion of the theorems follows from an application of the LLLL, once an appropriate p–lopsided graph, for a suitable chosen p, is defined. The construction of the p–lopsided graph is analogous in all the cases, but their analysis differs considerably in each of them and uses an elegant switching argument.

In the case of Theorem 1.2 we follow the same proof strategy as the one in [8]. However, it is not clear to us that a uniform bound on the multiplicity of edges can be treated with the same tools.

The case of a d-regular bipartite graph G requires a more careful analysis, since the uniform probability space on the perfect matchings of G is not a product space as for complete graphs. We believe that it is possible to extend our result to sparser regular bipartite graphs, provided that every color does not appear too many times on the edges of the graph.

Finally, the case of complete r–partite r–uniform hypergraphs is technically more involved due to the higher degree of complexity of the hypergraph structure, and all steps in the proof must be adapted to this setting.

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