The achromatic number of Kneser graphs

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Abstract

The achromatic number $\alpha$ of a graph is the largest number of colors that can be assigned to its vertices such that adjacent vertices have different color and every pair of different colors appears on the end vertices of some edge.

We estimate the achromatic number of Kneser graphs $K(n,k)$ and determine $\alpha(K(n,k))$ for some values of $n$ and $k$. Furthermore, we study the achromatic number of some geometric type Kneser graphs.

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1 Introduction

Let $G$ be a finite simple graph. An $l$-coloring of $G$ is a surjective function $\varsigma$ that assigns a number from the set $\{1, 2, \ldots, l\}$ to each vertex of $G$ such that any two adjacent vertices have different colors. An $l$-coloring $\varsigma$ is called complete if for each pair of different colors $i, j \in \{1, 2, \ldots, l\}$ there exists an edge $xy \in E(G)$ such that $\varsigma(x) = i$ and $\varsigma(y) = j$.

While the chromatic number $\chi(G)$ of $G$ is defined as the smallest number $l$ for which there exists an $l$-coloring of $G$, the achromatic number $\alpha(G)$ of $G$ is defined as the largest number $l$ for which there exists a complete $l$-coloring of $G$ (see [7]). Note that any $\chi(G)$-coloring of $G$ is also complete. Therefore, for any graph $G$

$$\chi(G) \leq \alpha(G).$$

Let $V$ be the set of all $k$-subsets of $\{1, 2, \ldots, n\}$, where $1 \leq k \leq n/2$. The Kneser graph $K(n, k)$ is the graph with vertex set $V$ such that two vertices are adjacent if and only if the corresponding subsets are disjoint. It is well-known that $\chi(K(n, k)) = n - 2(k - 1)$ (see [5]).

A complete geometric graph of $n$ points is a drawn of the complete graph $K_n$ in the plane such that its vertices is the set $P$ of points in general position, and its edges are straight-line segments connecting every pair of points in $P$. In [2], it was studied the chromatic number of graphs $D_P(n)$ whose vertex set is the set of edges of a complete geometric graph of $n$ points and adjacency is defined in terms of geometric disjointness.

The remainder of this paper is organized as follows: In Section 2, we estimate bounds for the achromatic number of Kneser graphs. In Section 3, we determine $\alpha(K(n, 2))$ for every $n$. Finally, in Section 4, we study the achromatic number of graphs $D_P(n)$.

2 Bounds for $\alpha(K(n, k))$

In this section, we prove general lower and upper bounds for the achromatic number of Kneser graphs.

2.1 Upper bounds

The following upper bound for the achromatic number was proved in a particular case in [1].
Theorem 2.1 Let $G$ be a graph of order $p$ with clique number $\omega(G)$, then

$$\alpha(G) \leq \left\lfloor \frac{p + \omega(G)}{2} \right\rfloor.$$  

Proof. Any complete coloring of $G$ has at most $\omega(G)$ classes of a single vertex. Then $\alpha(G) \leq \left\lfloor \frac{p - \omega(G)}{2} + \omega(G) \right\rfloor$ and the result follows. \qed

Since the clique number of a Kneser graph $K(n, k)$ is $\lfloor n/k \rfloor$ we have the following corollary.

Corollary 2.2 For any $n \geq 2$, $\alpha(K(n, k)) \leq \left\lfloor \frac{n^2 + \frac{n}{2}}{2} \right\rfloor \in O\left(\frac{n^k}{k!}\right)$.

The complement of the line graph of the complete graph on $n$ vertices is the Kneser graph $K(n, 2)$. In this case, we have the following upper bound.

Theorem 2.3 For any $n \geq 2$, $\alpha(K(n, 2)) \leq \left\lfloor \frac{n + 1}{2} \right\rfloor$.

Proof. Let $\varsigma$ be a complete coloring in $K(n, 2)$. Consider the graph $K(n, 2)$ as the complete graph $K_n$ such that disjoint edges of $K_n$ are adjacent vertices in $K(n, 2)$, then a chromatic class of size 2 in $\varsigma$ has to be a path of 3 vertices $xyz$ -- a $P_3$ subgraph --, then the vertex $y$ is not a vertex of an edge $e$ such that $\{e\}$ is a chromatic class of size 1. Therefore, any complete coloring of $K(n, 2)$ has at most $x$ classes of order 1 -- a matching in $K_n$; and at most $n - 2x$ classes of order 2 -- a set of pairwise disjoint $P_3$ subgraphs in $K_n$. Hence,

$$\alpha(K(n, 2)) \leq \left\lfloor \frac{n^2 - x - 2(n - 2x)}{3} + x + (n - 2x) \right\rfloor$$

then

$$\alpha(K(n, 2)) \leq \left\lfloor \frac{n^2 + 2x + (n - 2x)}{3} \right\rfloor = \left\lfloor \frac{n^2 + n}{3} \right\rfloor$$

and the result follows. \qed

2.2 General lower bound

An $l$-coloring $\varsigma$ is called dominating if every color class contains a vertex that has a neighbor in every other color class. The $b$-chromatic number $\varphi(G)$ of $G$ is defined as the largest number $l$ for which there exists a dominating $l$-coloring of $G$ (see [4]). Since a dominating coloring is also complete, hence, for any graph $G$

$$\chi(G) \leq \varphi(G) \leq \alpha(G).$$
The following theorem was proved in [6]:

**Theorem 2.4 (Hajiabolhassan [6])** Let $k \geq 3$ an integer. If $n \geq 2k$, then

\[ 2\left(\left\lfloor \frac{n}{2} \right\rfloor \right) \leq \varphi(K(n,k)). \]

In consequence, we have that $\alpha(K(n,k)) \in \Theta(n^k)$.  

**Corollary 2.5** For any $n \geq 2k \geq 6$, we have that

\[ \alpha(K(n,k)) \geq 2\left(\left\lfloor \frac{n}{2} \right\rfloor \right) \in \Omega\left(\frac{n^k}{2^{k-1}k!}\right). \]

In the following section is proved a better lower bound of $\alpha(K(n,2))$.

## 3 Tight lower bound of $\alpha(K(n,2))$

In this section, we prove that the bound stated in Theorem 2.3 is tight. We prove that $\alpha(K(n,2)) = \left\lfloor \frac{(n+1)}{3} \right\rfloor$ for every $n$. To derive the lower bound, we exhibit a complete coloring for $K(n,2)$. To simplify the proof of the main statement of this section, consider the graph $K(n,2)$ as the complete graph $K_n$ such that disjoint edges of $K_n$ are adjacent vertices in $K(n,2)$. Now, we define the following known concepts.

A **Steiner triple system** $STS(n+1)$ is a pair $(P, B)$ where $P$ is a $(n+1)$-set of points and $B$ is a collection of 3-subsets of $P$ called blocks. Each 2-subset of $P$ appears in precisely one block and each point of $P$ appears in precisely $n/2$ blocks. It is well-known that a $STS(n+1)$ there exists if and only if $n+1 \equiv 1, 3 \mod 6$ (see [8]). A $STS(n+1)$ can naturally be regarded as an edge-partition (into triangles) of the complete graph $K_{n+1}$.

Theorem 3.2 is proved by cases. The simplest case is given in the following lemma.

**Lemma 3.1** If $n + 1 \equiv 1, 3 \mod 6$ then

\[ \frac{1}{3}\left(\frac{n+1}{2}\right) \leq \alpha(K(n,2)). \]

**Proof.** Let $n + 1 \equiv 1, 3 \mod 6$, then there exists $STS(n+1)$. Consider $STS(n+1)-x$ where $x$ is a point of $STS(n+1)$ in the natural relation with $K_n$, therefore, we have an edge partition into triangles and a perfect matching. Putting a different color in every triangle and in every edge in the perfect matching we get a $(n+1)/3$-coloring which is complete since if two color classes have a point in common then always there are two no adjacent edges.\(\square\)
We omit the proof of the remaining cases, namely, when \( n \equiv 1, 3, 4, 5 \mod 6 \).

**Theorem 3.2** Let \( n \) a natural number then
\[
\alpha(K(n, 2)) = \left\lfloor \frac{1}{3} \binom{n+1}{2} \right\rfloor.
\]

4 The achromatic number of \( D(n) \)

Let \( P \) be a set of \( n \) points in general position in the plane, i.e., no three points are collinear. Let \( D_P(n) \) be the graph whose vertex set is the set of all straight-line segments with end points in \( P \) such that every two disjoint segments are adjacent. Notice that each graph \( D_P(n) \) is a spanning subgraph of \( K(n, 2) \). The chromatic number of graphs \( D_P(n) \) was first studied in [2] where the authors prove that
\[
2 \left\lfloor \frac{n+1}{3} \right\rfloor - 1 \leq \chi(D_P(n)) \leq \min\{n - 2, n + \frac{1}{2} - \frac{\log\log(n)}{2}\}.
\]

In this section, we prove bounds for \( \alpha(D_P(n)) \). To begin with, we obtain the following upper bound using Theorem 2.1.

Since there are point subsets of \( P \) for which \( \frac{n}{2} \) edges are pairwise disjoint, then, we have the following.

**Corollary 4.1** For any \( n \geq 2 \) and any point set \( P \) in general position,
\[
\alpha(D_P(n)) \leq \left\lfloor \frac{\binom{n}{2} + \frac{n}{2}}{2} \right\rfloor \leq \frac{n^2}{4}.
\]

To achieve the lower bound, we prove the following lemma. Recall that a *straight-line thrackle* (for short, *thrackle*) is a set of straight-line edges such that any two distinct edges either meet at exactly one common point or they cross.

**Lemma 4.2** Any two triangles \( T_1 \) and \( T_2 \) with points in \( P \), that share at most one point, contain two disjoint edges.

**Proof.** Case 1. \( T_1 \) has a point in common with \( T_2 \): It is known that every \((r + 1)\)-edge set on an \( r \)-point set has at least two disjoint edges (see [3]). In particular when \( r = 5 \).

Case 2. \( T_1 \) has no points in common with \( T_2 \): Let \( e \) be an edge of \( T_2 \). Therefore, \( e \) and \( T_1 \) is a thrackle which is impossible (see [3]).

\( \square \)
If we identify a Steiner triple system $STS(n)$ with the complete geometric graph of $n$ points and we color each triangle with a different color, by Lemma 4.2, we have the following.

**Lemma 4.3** If $n \equiv 1, 3 \mod 6$ and $P$ is a point set in general position, then

$$\frac{n^2}{6} - \frac{n}{6} = \frac{1}{3} \binom{n}{2} \leq \alpha(D_P(n)).$$

Finally, we have the following theorem.

**Theorem 4.4** For any natural number $n$ any point set $P$ in general position,

$$\frac{n^2}{6} - \Theta(n) \leq \alpha(D_P(n)) \leq \frac{n^2}{4}.$$

**References**


