Homomorphisms of Strongly Regular Graphs

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Abstract
We prove that if $G$ and $H$ are primitive strongly regular graphs with the same parameters and $\varphi$ is a homomorphism from $G$ to $H$, then $\varphi$ is either an isomorphism or a coloring (homomorphism to a complete subgraph). Moreover, any such coloring is optimal for $G$ and its image is a maximum clique of $H$. Therefore, the only endomorphisms of a primitive strongly regular graph are automorphisms or colorings. This confirms and strengthens a conjecture of Peter Cameron and Priscila Kazanidis that all strongly regular graphs are cores or have complete cores. The proof of the result is elementary, mainly relying on linear algebraic techniques.

Keywords: graph homomorphisms, strongly regular graphs, Lovász theta, cores.

1 Introduction

A homomorphism between two graphs $G$ and $H$ is a function $\varphi : V(G) \to V(H)$ such that $\varphi(u) \sim \varphi(v)$ whenever $u \sim v$, where $\sim$ denotes adjacency. Whenever a homomorphism exists from $G$ to $H$, we write $G \to H$, and if

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$^1$ Thanks to Chris Godsil, Krystal Guo, Laura Mančinska, Brendan Rooney, Robert Šámal, and Antonis Varvitsiotis. The author is supported by the EPSRC.

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both $G \rightarrow H$ and $H \rightarrow G$ then we say that $G$ and $H$ are homomorphically equivalent. Given a homomorphism $\varphi$ from $G$ to $H$, we will refer to the subgraph of $H$ induced by $\{\varphi(u) : u \in V(G)\}$ as the image of $\varphi$. It is easy to see that a $c$-coloring of a graph $G$ is equivalent to a homomorphism from $G$ to the complete graph on $c$ vertices, $K_c$. More generally, we will refer to any homomorphism whose image is a clique (complete subgraph) as a coloring.

A homomorphism from a graph $G$ to itself is called an endomorphism, and it is said to be proper if it is not an automorphism of $G$, or equivalently, its image is a proper subgraph of $G$. A graph with no proper endomorphisms is said to be a core, and these play a fundamental role in the theory of homomorphisms since every graph is homomorphically equivalent to a unique core. We refer to the unique core homomorphically equivalent to $G$ as the core of $G$. It is known [3], and not difficult to show, that the core of $G$ is isomorphic to any vertex minimal induced subgraph of $G$ to which $G$ admits an endomorphism.

If the core of a graph $G$ is a complete graph $K_c$, then $G$ must contain a clique of size $c$ and must also be $c$-colorable. Therefore, $\omega(G) = \chi(G) = c$, where $\omega(G)$ and $\chi(G)$ are the clique and chromatic numbers of $G$ respectively. Conversely, if $\omega(G) = \chi(G) = c$, then the core of $G$ is $K_c$. If a graph is either a core or has a complete graph as a core, then it is said to be core-complete. Many known results on cores are statements saying that all graphs in a certain class are core-complete [1,2,4], and often it remains difficult to determine whether a given graph in the class is a core or has a complete core.

For some classes of graphs, something stronger than core-completeness can be shown. A graph $G$ is a pseudocore if every proper endomorphism of $G$ is a coloring. It follows that any pseudocore is core-complete, but the converse does not hold (consider a complete multipartite graph). Similarly, it is easy to see that any core is a pseudocore, but the converse does not hold in this case either, e.g. the Cartesian product of two complete graphs of size at least three.

In this work, we will focus on homomorphisms and cores of strongly regular graphs. An $n$-vertex $k$-regular graph is said to be strongly regular with parameters $(n, k, \lambda, \mu)$ if every pair of adjacent vertices has $\lambda$ common neighbors, and every pair of distinct non-adjacent vertices has $\mu$ common neighbors. For short, we will call such a graph an $\text{SRG}(n, k, \lambda, \mu)$. A strongly regular graph is called imprimitive if either it or its complement is disconnected. In such a case, the graph or its complement is a disjoint union of equal sized complete graphs. Homomorphisms of these graphs are straightforward, and so we will only consider primitive strongly regular graphs here. Because of this, from now on when we consider a strongly regular graph, we will implicitly assume
that it is primitive. In this case, we always have that $1 \leq \mu < k$, and that the diameter is two.

In [1], Cameron and Kazanidis showed that rank 3 graphs are core-complete. A graph is rank 3 if its automorphism group acts transitively on vertices, ordered pairs of adjacent vertices, and ordered pairs of distinct non-adjacent vertices. It is not hard to see that any rank 3 graph is strongly regular, and in fact the latter are often viewed as combinatorial relaxations of the former. Because of this, Cameron and Kazanidis conjectured that strongly regular graphs are core-complete as well. We confirm and strengthen this conjecture by proving the following result: if $G$ and $H$ are both primitive $SRG(n, k, \lambda, \mu)$’s and $\varphi$ is a homomorphism from $G$ to $H$, then $\varphi$ is either an isomorphism or a coloring. Letting $G = H$, this statement implies that all strongly regular graphs are pseudocores. From this and previously known results it follows that a homomorphism that is not an isomorphism exists between $G$ and $H$ if and only if $\chi(G) = \omega(H)$. Moreover, in this case the common value of $\chi(G)$ and $\omega(H)$ is equal to the Hoffman bound on chromatic number which depends only on $(n, k, \lambda, \mu)$. From here we see that any strongly regular graph $G$ falls into one of four classes depending what subset of $\{\omega(G), \chi(G)\}$ meets the Hoffman bound. Using this we show that the homomorphism order of strongly regular graphs with a fixed parameter set has a simple description.

Although the main concrete contribution of this work is the resolution of the Cameron and Kazanidis conjecture, we believe that the real significance is the step taken towards understanding how combinatorial regularity, as opposed to symmetry, can impact the endomorphisms and core of a graph.

## 2 Homomorphism Matrices

The adjacency matrix of any primitive strongly regular graph has exactly three eigenvalues that depend only on its parameters, the largest being the valency $k$, and the others usually referred to as $\theta$ and $\tau$ where $\theta > 0 > \tau$. Suppose $G$ and $H$ are strongly regular graphs with the same parameter set and $\theta$ is their common second eigenvalue. To any homomorphism from $G$ to $H$, we will associate a matrix $X$ with rows and columns indexed by $V(G)$ and defined entrywise as follows:

$$X_{uv} = \begin{cases} 
\theta & \text{if } u \not\sim v \text{ and } \varphi(u) = \varphi(v) \\
-1 & \text{if } u \not\sim v \text{ and } \varphi(u) \sim \varphi(v) \\
0 & \text{o.w.}
\end{cases}$$
Note that \( u \not\equiv v \) means that \( u \) and \( v \) are distinct non-adjacent vertices. We will also say that \( u \) and \( v \) are non-neighbors when \( u \not\equiv v \). The matrix \( X \) will be called the homomorphism matrix of \( \varphi \). Essentially, the homomorphism matrix of \( \varphi \) keeps track of the vertices whose “relationship” changes under the mapping \( \varphi \). The following property of the homomorphism matrix is the key to proving our main result.

**Lemma 2.1** Let \( G \) and \( H \) both be \( \mathrm{SRG}(n, k, \lambda, \mu) \)'s and let \( \varphi \) be a homomorphism from \( G \) to \( H \). Also let \( A \) be the adjacency matrix of \( G \), and \( \tau \) its least eigenvalue. If \( X \) is the homomorphism matrix of \( \varphi \), then \( X(A - \tau I) = 0 \).

The above property of homomorphism matrices actually follows from the same holding for two other matrices we define below:

\[
\begin{align*}
\left( \hat{E}_\tau \right)_{uv} &= \begin{cases} 
1 & \text{if } u = v \\
\tau/k & \text{if } u \sim v \\
\bar{\theta}/k & \text{if } u \not\sim v
\end{cases} \\
M^\varphi_{uv} &= \begin{cases} 
1 & \text{if } \varphi(u) = \varphi(v) \\
\tau/k & \text{if } \varphi(u) \sim \varphi(v) \\
\bar{\theta}/k & \text{if } \varphi(u) \not\sim \varphi(v)
\end{cases}
\end{align*}
\]

where \( \bar{k} = n-k-1 \) and \( \bar{\theta} = -\tau-1 \) are the valency and second eigenvalue of the complement of \( G \). The matrix \( \hat{E}_\tau \) is actually a positive scalar multiple of the projection onto the \( \tau \)-eigenspace of \( G \). Therefore, \( \hat{E}_\tau \) is positive semidefinite and \( \hat{E}_\tau(A - \tau I) = 0 \). The \( uv \)-entry of \( M^\varphi \) is equal to the \( \varphi(u)\varphi(v) \)-entry of \( \hat{F}_\tau \), a scalar multiple of the projection onto the \( \tau \)-eigenspace of \( H \). From here it can be shown that \( M^\varphi \) is positive semidefinite. Furthermore, one can show that \( \text{Tr}(M^\varphi(A - \tau I)) = \text{Tr}(\hat{E}_\tau(A - \tau I)) = 0 \), and thus \( M^\varphi(A - \tau I) = 0 \) since they are both psd. Finally, from an identity relating parameters of SRGs, it follows that the homomorphism matrix \( X \) is a scalar multiple of \( M^\varphi - \hat{E}_\tau \).

### 3 Properties of Homomorphisms Between SRGs

Here we see the significance of Lemma 2.1. Specifically, since \( X(A - \tau I) = 0 \) for any homomorphism matrix \( X \), we can consider vertices \( u, v \in V(G) \) and unravel the combinatorial meaning of \( (X(A - \tau I))_{uv} = 0 \) to determine properties of the underlying homomorphism. This is the key insight of this work.

First we will need some notation. Given \( G \) and \( H \) that are \( \mathrm{SRG}(n, k, \lambda, \mu) \)'s and a homomorphism \( \varphi \) from \( G \) to \( H \), we define the following for \( u, u' \in V(G) \):

- \( N_u = \{ v \in V(G) : v \sim u \} \);
- \( D_u = \{ v \in V(G) : v \not\equiv u \& \varphi(v) \sim \varphi(u) \} \);
- \( F_u = \varphi^{-1}(\varphi(u)) = \{ v \in V(G) : v \not\equiv u \& \varphi(v) = \varphi(u) \} \).
In words, $N_u$ is the set of neighbors of $u$, the set $D_u$ contains the non-neighbors of $u$ that become neighbors after applying $\varphi$ (to both $u$ and the non-neighbors), and $F_u$ is simply the fibre (preimage of a single vertex) of $\varphi$ containing $u$.

It is worth giving the entries of a homomorphism matrix $X$ in terms of the above notation in order to make the subsequent proofs more clear:

$$X_{uv} = \begin{cases} 
\theta & \text{if } v \in F_u \setminus \{u\} \\
-1 & \text{if } v \in D_u \\
0 & \text{otherwise}
\end{cases}$$

**Lemma 3.1** Let $G$ and $H$ both be $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. Then $|D_u| = \theta(|F_u| - 1)$.

**Proof.** Let $X$ be the homomorphism matrix of $\varphi$. The $u$-row of $X$ lies in the $\tau$-eigenspace of $G$. Since the all ones vector is an eigenvector of $G$ with eigenvalue $k$, we have that the sum of the entries of the $u$-row of $X$ is zero. Therefore, $(|F_u| - 1)\theta - |D_u| = 0$. \hfill $\square$

**Lemma 3.2** Let $G$ and $H$ both be $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. If $\varphi(u) \not\sim \varphi(u')$, then for all $v \in D_u$ we have $v \not\sim u'$.

**Proof.** Computing the $uu'$-entry of $X(A - \tau I)$ for $u, u'$ as described in the lemma statement, we see that $0 = (X(A - \tau I))_{uu'} = -|D_u \cap N_{u'}|$. \hfill $\square$

Continuing onward, we can also obtain the following lemmas:

**Lemma 3.3** Let $G$ and $H$ both be $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. Suppose that $\varphi(u) \not\sim \varphi(u')$. If $v$ is a common neighbor of $u$ and $u'$, then $v$ is adjacent to every vertex of $F_u \cup F_{u'}$. Moreover, if $v_1$ and $v_2$ are any two distinct vertices in $F_u \cup F_{u'}$, then $N_{v_1} \cap N_{v_2} = N_u \cap N_{u'}$.

**Lemma 3.4** Let $G$ and $H$ both be $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. Suppose that $\varphi(u) \not\sim \varphi(u')$ and let $v$ be a common neighbor of $u$ and $u'$. If $x$ is a common non-neighbor of $u$ and $v$, then $\varphi(x) \not\sim \varphi(u)$.

**Lemma 3.5** Let $G$ and $H$ both be primitive $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. If $u, u' \in V(G)$ and $\varphi(u) \not\sim \varphi(u')$, then $|F_u| = 1$.

### 4 Main Result

Here we combine the results of the previous section to prove our main result.

**Theorem 4.1** Let $G$ and $H$ both be primitive $\text{SRG}(n,k,\lambda,\mu)$’s and let $\varphi$ be a homomorphism from $G$ to $H$. Then $\varphi$ is a coloring or an isomorphism.
Proof. We first show that if $|F_u| = 1$, then $|F_v| = 1$ for all $v \not\cong u$. Suppose that $|F_u| = 1$. Then by Lemma 3.1, we have that $|D_u| = \theta(|F_u| - 1) = 0$ and thus $D_u = \emptyset$. Since $|F_u| = 1$ also implies that $F_u \setminus \{u\} = \emptyset$, we have that $\varphi(v) \not\cong \varphi(u)$ for all $v \not\cong u$. Applying Lemma 3.5, we obtain $|F_v| = 1$ for all $v \not\cong u$ and thus we have proven the claim.

Now we prove the theorem. If $\varphi$ is a coloring then we are done, so suppose that it is not. This is equivalent to there being $u, u' \in V(G)$ such that $\varphi(u) \not\cong \varphi(u')$. Applying Lemma 3.5 again, we see that $|F_u| = 1$. By the above claim and the fact that $G$ is connected, we have that $|F_v| = 1$ for all $v \in V(G)$, i.e., that $\varphi$ is injective. Since $G$ and $H$ have the same number of vertices and edges, this proves that $\varphi$ is an isomorphism. \(\square\)

As a corollary we obtain the following strengthening of the Cameron and Kazanidis conjecture:

**Corollary 4.2** Every primitive strongly regular graph is a pseudocore.

We can combine Theorem 4.1 with previously known results on strongly regular graphs to obtain the following:

**Lemma 4.3** Let $G$ and $H$ both be primitive $\text{SRG}(n, k, \lambda, \mu)$’s with least eigenvalue $\tau$. Then there exists a homomorphism that is not an isomorphism from $G$ to $H$ if and only if $\chi(G) = 1 \frac{k}{\tau} = \omega(H)$.

The above lemma suggests a useful partition of strongly regular graphs with a fixed parameter set. Namely, to partition them into four types according to what subset of $\{\omega(G), \chi(G)\}$ is equal to $1 \frac{k}{\tau}$. The existence of homomorphisms between non-isomorphic $\text{SRG}(n, k, \lambda, \mu)$’s is then completely determined by type, and thus the homomorphism order of $\text{SRG}(n, k, \lambda, \mu)$’s can be described succinctly.

**References**


