The Orthogonal Art Gallery Theorem with Constrained Guards

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Abstract
Let $P$ be an orthogonal polygon with $n$ vertices, and let $V^*$ and $E^*$ be specified sets of vertices and edges of $P$. We prove that $P$ has a guard set of cardinality at most $\left\lfloor (n + 3|V^*| + 2|E^*|) / 4 \right\rfloor$ that includes each vertex in $V^*$ and at least one point of each edge in $E^*$. Our bound is sharp and reduces to the orthogonal art gallery theorem of Kahn, Klawe and Kleitman when $V^*$ and $E^*$ are empty.

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1 Introduction: Art Galleries with Constrained Guards

The original art gallery problem, posed by Klee in 1973, asks for the minimum number of guards (points) sufficient to protect any simple polygon (the art gallery). Every point inside the polygon must be visible to at least one guard, i.e., the segment joining the point to some guard must not intersect the exterior of the polygon. The first solution to this problem was given by Chvátal [1], who proved that \( \lfloor n/3 \rfloor \) guards are sometimes necessary, and always sufficient to cover a polygon with \( n \) vertices. Later Fisk [2] provided a shorter proof of Chvátal’s theorem using an elegant graph coloring argument. Klee’s art gallery problem has since grown into a significant area of study. Numerous variants of Klee’s art gallery problem have been proposed and studied with different restrictions placed on the shape of the galleries or the powers of the guards. (See the monograph by O’Rourke [6], and the surveys by Shermer [8] and Urrutia [7].)

Some of the most important variants of Klee’s art gallery problem deal with orthogonal polygons — those whose interior angles are 90° or 270°. In 1983, Kahn, Klawe and Kleitman [3] proved the fundamental theorem about orthogonal art galleries:

**Theorem 1.1 ([3])** An orthogonal art gallery with \( n \) vertices can be protected by \( \lfloor n/4 \rfloor \) vertex guards.

A very recent variant of Klee’s art gallery problem [5] deals with constraints on the placement of the guards in the triangulation graph \( G = (V, E) \). Each vertex of a specified subset \( V^* \) must be a guard, and each edge of a specified edge set \( E^* \) must contain a guard. Here is the main theorem of [5]:

**Theorem 1.2 ([5])** Let \( P \) be a polygon with \( n \) vertices. If \( V^* \) and \( E^* \) are specified vertex and edge subsets of \( P \), then \( P \) has a guard set of cardinality at most

\[
\left\lfloor \frac{n + 2|V^*| + |E^*|}{3} \right\rfloor
\]

that includes each vertex in \( V^* \) and at least one point from each edge in \( E^* \).

The ordinary art gallery theorem is the special case in which \( V^* \) and \( E^* \) are empty.

Our main theorem establishes the corresponding result for orthogonal polygons:

**Theorem 1.3** Let \( P \) be an orthogonal polygon with \( n \) vertices. If \( V^* \) and \( E^* \) are specified vertex and edge subsets of \( P \), then \( P \) has a guard set of cardinality
Fig. 1. An orthogonal 20-gon with $|V^*| = 5$ and $|E^*| = 3$ that requires 10 guards. Asterisks indicate specified vertices and edges.

at most

$$\left\lfloor \frac{n + 3|V^*| + 2|E^*|}{4} \right\rfloor$$

that includes each vertex in $V^*$ and at least one point from each edge in $E^*$.

The orthogonal art gallery theorem of [3] is the special case in which $V^*$ and $E^*$ are empty. The bound from Theorem 1.3 is sharp. Configurations that achieve equality have a sequence of $|V^*|$ consecutive specified vertices followed by a sequence of $|E^*|$ specified edges alternating with unspecified edges, as depicted in Figure 1.

In Section 2 we prove Theorem 1.3 using a graph coloring argument similar to Fisk’s [2]. In Section 3 we state an algorithm that constructs a constrained guard set satisfying Theorem 1.3. The correctness of the algorithm can be established by induction in a scheme similar to the one in [5], and we omit the details of the proof. (The complexity of algorithms arising from our proofs is $O(n \log n)$, from the convex quadrangulation [4].)

## 2 A Graph Coloring Argument

The difficult part in the proof of Theorem 1.1 was establishing the existence of a convex quadrangulation, now a fundamental tool in the field:

Proposition 2.1 ([3]) An orthogonal polygon can be partitioned into convex quadrilaterals by inserting suitable non-crossing diagonals.

In this section we use convex quadrangulations to prove a somewhat stronger result than Theorem 1.3. If $G = (V, E)$ is a quadrangulation graph of a polygon, then we say that the vertex subset $G$ is a guard set for $G$ provided every quadrilateral has a vertex in $G$. Clearly, a guard set for a convex quadrangulation graph is also a guard set for the polygon.
Proposition 2.2 Let \( P \) be a polygon that admits a convex quadrangulation and let \( G = (V, E) \) be the corresponding quadrangulation graph. If \( V^* \) and \( E^* \) are specified vertex and edge subsets of \( G \), then \( G \) has a guard set of cardinality at most

\[
\left\lfloor \frac{n + 3|V^*| + 2|E^*|}{4} \right\rfloor
\]

that includes each vertex in \( V^* \) and at least one end-vertex of each edge in \( E^* \).

Proof. It is easy to see that the convex quadrangulation graph \( G = (V, E) \) admits a proper 4-coloring with colors 1, 2, 3, and 4, such that each quadrilateral has one vertex of each color. Let \( X_i \) be the set of vertices of color \( i \). We will augment each \( X_i \) so that it satisfies our constraints on the vertices and edges. Let \( Y_i \) be the set of vertices in \( V^* \) of color \( i \). Note that

\[
|Y_1| + |Y_2| + |Y_3| + |Y_4| = |V^*|.
\]

Partition the set \( E^* \) of specified edges into six sets \( E^*_{ij} \), \( 1 \leq i < j \leq 4 \) where \( E^*_{ij} \) is the set of edges in \( E^* \) whose end-vertices are colors \( i \) and \( j \). Let \( Z_{ij} \) be the set of end-vertices of color \( i \) among the edges in \( E^*_{ij} \), where \( i < j \).

Note that

\[
|Z_{12}| + |Z_{13}| + |Z_{14}| + |Z_{23}| + |Z_{24}| + |Z_{34}| = |E^*|.
\]

Each of the four sets

\[
X_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Z_{23} \cup Z_{24} \cup Z_{34}, \quad X_2 \cup Y_1 \cup Y_3 \cup Y_4 \cup Z_{13} \cup Z_{14} \cup Z_{34},
\]
\[
X_3 \cup Y_1 \cup Y_2 \cup Y_4 \cup Z_{12} \cup Z_{14} \cup Z_{24}, \quad X_4 \cup Y_1 \cup Y_2 \cup Y_3 \cup Z_{12} \cup Z_{13} \cup Z_{23}
\]

is a constrained guard set for \( G \). The sum of the cardinalities of these four sets is no more than \( n + 3|V^*| + 2|E^*| \). It follows that at least one of the constrained guard sets has cardinality at most \( \lfloor (n + 3|V^*| + 2|E^*|)/4 \rfloor \). \( \Box \)

3 An Algorithm to Construct a Constrained Guard Set

We give a recursive algorithm that constructs a constrained guard set satisfying Theorem 1.3.

Input: a convex quadrangulation of a polygon \( P_n \) with \( n \) vertices, a specified vertex set \( V^* \), and a specified edge set \( E^* \).

Output: a guard set \( G(P_n, V^*, E^*) \).

Without loss of generality, no vertex in \( V^* \) is incident to an edge in \( E^* \), and no two edges in \( E^* \) share a vertex. Indeed, if two edges \( e_1 \) and \( e_2 \) from \( E^* \) share a vertex \( v \), then define \( G(P_n, V^*, E^*) = G(P_n, V^* \cup \{v\}, E^* \setminus \{e_1, e_2\}) \). If one vertex \( v \) from \( V^* \) is incident to an edge \( e \) from \( E^* \), then define \( G(P_n, V^*, E^*) = G(P_n, V^*, E^* \setminus \{e\}) \).
Suppose that \( n = 4 \). If \( V^* \) and \( E^* \) are both empty, then let \( G(P_4, V^*, E^*) = \{ v \} \), where \( v \) is any vertex of \( P_4 \). Otherwise, place guards at each element of \( V^* \) and at one end-vertex of each edge in \( E^* \) to construct \( G(P_4, V^*, E^*) \).

Suppose that \( n \geq 6 \). Let \( x_1x_2 \) be a diagonal of \( P_n \) that splits \( P_n \) into the quadrilateral \( x_1x_2y_2y_1 \) and the polygon \( P_{n-2} \), as in Figure 2. Let \( V_1^* = V^* \cap \{ y_1, y_2 \} \), and let \( E_1^* = E^* \cap \{ x_1y_1, y_1y_2, x_2y_2 \} \).

Case 1: \( V_1^* = \emptyset \) and \( E_1^* = \emptyset \).
Define \( G(P_n, V^*, E^*) = G(P_{n-2}, V^*, E^* \cup \{ x_1, x_2 \}) \).

Case 2: \( V_1^* \neq \emptyset \) and \( E_1^* = \emptyset \).
Define \( G(P_n, V^*, E^*) = G(P_{n-2}, V^* - V_1^*, E^*) \cup V_1^* \).

Case 3: \( V_1^* = \emptyset \) and \( E_1^* \neq \emptyset \).
If \( y_1y_2 \in E^* \), then define \( G(P_n, V^*, E^*) = G(P_{n-2}, V^*, E^* - E_1^*) \cup \{ y_1 \} \). Otherwise, \( G(P_n, V^*, E^*) = G(P_{n-2}, V^* \cup V_0^*, E^* - E_1^*) \), where \( V_0^* \) is the intersection of the set of the end-vertices of \( E_1^* \) and \( \{ x_1, x_2 \} \).

Case 4: \( V_1^* \neq \emptyset \) and \( E_1^* \neq \emptyset \).
Then each of \( V_1^* \) and \( E_1^* \) must have exactly one element. Let \( y_i \) be the unique element of \( V_1^* \), and let \( x_jy_j \) be the unique element of \( E_1^* \). Define \( G(P_n, V^*, E^*) = G(P_{n-2}, V^* - V_1^* \cup \{ x_j \}, E^* - E_1^*) \cup \{ y_i \} \).

It is easy to prove the correctness of the algorithm in a manner similar to [5]:

Proposition 3.1 The set \( G(P_n, V^*, E^*) \) constructed by the algorithm is a guard set for \( P_n \) that includes each vertex in \( V^* \) and at least one end-vertex of
each edge in \( E^* \). Moreover,

\[
|G(P_n, V^*, E^*)| \leq \left\lfloor \frac{n + 3|V^*| + 2|E^*|}{4} \right\rfloor.
\]

Theorem 1.3 follows from Propositions 3.1 and 2.1.

References


