On the Construction of Triples of Diagonal Latin Squares of Order 10

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Abstract

We provide a triple of diagonal Latin squares of order 10 that is the closest to being a triple of mutually orthogonal diagonal Latin squares found so far. It was obtained by constructing all orthogonal mates for diagonal Latin squares generated according to a specific scheme. We also show that a triple of mutually orthogonal diagonal

Latin squares of order 10 cannot be constructed based on diagonal Latin squares from specific families.

Keywords: Latin square, diagonality, orthogonality, SAT.

1 Introduction

A Latin square $A = (a_{ij})$ of order n is an $n \times n$ table filled with symbols from the set $N = \{0, 1, ..., n-1\}$, in such a way that each symbol occurs precisely once in each row and each column. Diagonal Latin square is a Latin square, in which each symbol from N occurs precisely once in its main diagonal and in its main antidiagonal. Two Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ are orthogonal if all ordered pairs (a_{ij}, b_{ij}) are distinct. A set of Latin squares, each two of them orthogonal, is called a set of mutually orthogonal Latin squares (MOLS). A set of diagonal Latin squares, each two of them orthogonal, is called a set of mutually orthogonal diagonal Latin squares (MODLS). A transversal of a Latin square is a set of n entries such that no two entries share the same row, column or symbol. Latin square A is isotopic to a Latin square B if B can be obtained from A by any of the following operations: permuting rows, permuting columns, permuting the names of the symbols.

Our research was mostly inspired by the papers [4] and [3]. In [4] it was determined that Latin squares of order 10 from several considered families cannot participate in a triple of MOLS of order 10. In [3] there was found the triple of Latin squares of order 10 that is the closest to being a triple of MOLS found so far. In this triple two out of three pairs of Latin squares are orthogonal, and for the third 91 pairs of elements out of 100 are distinct. We focused on obtaining similar results for diagonal Latin squares of order 10.

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2 Searching for Triples of Diagonal Latin Squares, in Which Orthogonality Condition is Partially Satisfied

In [2] it was shown, that pairs of MODLS of order 10 do exist. In particular, in that paper 3 pairs of this kind were presented. Interesting fact is that two out of three pairs from [2] share the same diagonal Latin square of order 10. It essentially means that there was found a triple of diagonal Latin squares such that in this triple two out of three pairs of diagonal Latin squares are orthogonal. For the third pair there are 60 different pairs of elements out of 100. For convenience let us refer to the number of different pairs of elements for such triples as *characteristics*. So, in this notation, in [2] there was suggested a triple of diagonal Latin squares of order 10 with characteristics of 60. We focused on searching for the triple of diagonal Latin squares of order 10 with characteristics more than 60.

We performed the search for a record triple in 3 stages. On the first stage a special set of diagonal Latin squares of order 10 is generated. On the second stage for each generated square all its orthogonal diagonal mates are found using the approach from [5]. According to this approach one constructs all transversals of a given Latin square, and then searches for all different sets of n disjoint transversals (where n is the order of a Latin square). In our case only diagonal orthogonal mates are required, so only transversals with exactly one element on the main diagonal and exactly one element on the main antidiagonal are used. On the third stage for each constructed diagonal Latin square, that has at least 2 orthogonal diagonal mates, all possible triples of diagonal Latin squares of the proposed kind are considered, and the one with the largest value of characteristics is chosen.

Let us consider the first stage in more detail. First some Latin square $A = (a_{ij})$ of order 5, such that each of its antidiagonals consists of one recurring element, is constructed. For each $k \in \mathbb{N}, k \in \{k_{min}, ..., k_{max}\}, k_{min} \geq 3, k_{max} \leq 25$, s iterations are performed (in practice we used $s = 1000$). In every iteration random k cells in A are selected. In each selected cell element a_{ij} is replaced by $9 - a_{ij}$, thus obtaining square B. Square C is formed by replacing each element b_{ij} in B by $9 - b_{ij}$. Square C' (B') is formed by reversing the order of columns in C (B). These 4 squares are used to form Latin square L of order 10 in the following way: $\left(\begin{smallmatrix} B & C' \\ C & B' \end{smallmatrix}\right)$. Finally diagonal Latin squares, which are isotopic to L (only rows permutation was used), are constructed. The result of the first stage is a summation of different diagonal Latin squares, constructed in all iterations for each k.

Using this approach we found a lot of triples with characteristics more than 60, including one with with the characteristics of 74 (it is shown below).

$$\begin{bmatrix} 0&1&2&3&4&5&6&7&8&9 \\ 1&2&3&4&9&0&5&6&7&8 \\ 4&0&8&7&6&3&2&1&9&5 \\ 9&8&7&6&5&4&3&2&1&0 \\ 5&9&1&2&3&6&7&8&0&4 \\ 3&5&9&8&2&7&1&0&4&6 \\ 2&3&4&0&8&1&9&5&6&7 \\ 7&6&5&9&1&8&0&4&3&2 \\ 6&4&0&1&7&2&8&9&5&3 \\ 8&7&6&5&0&9&4&3&2&1 \end{bmatrix} \begin{bmatrix} 0&1&2&3&4&5&6&7&8&9 \\ 8&7&5&6&0&9&3&2&1&4 \\ 1&4&6&0&5&8&9&3&2&7 \\ 3&9&4&8&1&2&7&5&0&6 \\ 6&5&9&1&2&4&8&0&7&3 \\ 9&2&8&7&6&3&4&1&5&0 \\ 4&0&7&5&3&6&1&8&9&2 \\ 5&3&0&4&7&1&2&9&6&8 \\ 7&8&3&2&9&0&5&6&4&1 \\ 2&6&1&9&8&7&0&4&3&5 \end{bmatrix} \begin{bmatrix} 0&1&2&3&4&5&6&7&8&9 \\ 9&4&8&7&6&3&0&1&2&5 \\ 5&7&1&6&9&2&3&8&0&4 \\ 7&9&5&8&3&6&1&0&4&2 \\ 1&2&6&9&7&0&4&3&5&8 \\ 4&8&3&0&5&9&2&6&1&7 \\ 8&6&0&4&2&7&5&9&3&1 \\ 3&5&7&1&0&4&8&2&9&6 \\ 2&3&9&5&8&1&7&4&6&0 \\ 6&0&4&2&1&8&9&5&7&3 \end{bmatrix}$$

The experiment took about 7 days on one computer. The following Latin square of order 5 (it is the addition table for Z_5) was used on the first stage.

$$\begin{matrix} 0&1&2&3&4 \\ 1&2&3&4&0 \\ 2&3&4&0&1 \\ 3&4&0&1&2 \\ 4&0&1&2&3 \end{matrix}$$

We used $k_{min} = 3, k_{max} = 11$. The most interesting result was obtained for $k = 8$. Below we show constructed B and C (in B values of the cells, selected for the transformation, are bold).

$$B = \begin{bmatrix} 0&1&2&3&4 \\ 1&2&3&4&\mathbf{9} \\ 2&3&4&0&\mathbf{8} \\ 3&\mathbf{5}&\mathbf{9}&\mathbf{8}&2 \\ 4&0&\mathbf{8}&\mathbf{7}&\mathbf{6} \end{bmatrix} \quad C = \begin{bmatrix} 9&8&7&6&5 \\ 8&7&6&5&0 \\ 7&6&5&9&1 \\ 6&4&0&1&7 \\ 5&9&1&2&3 \end{bmatrix}$$

Using B' and C' the following Latin square L of order 10 was constructed.

$$\begin{matrix} 0&1&2&3&4&5&6&7&8&9 \\ 1&2&3&4&9&0&5&6&7&8 \\ 2&3&4&0&8&1&9&5&6&7 \\ 3&5&9&8&2&7&1&0&4&6 \\ 4&0&8&7&6&3&2&1&9&5 \\ 9&8&7&6&5&4&3&2&1&0 \\ 8&7&6&5&0&9&4&3&2&1 \\ 7&6&5&9&1&8&0&4&3&2 \\ 6&4&0&1&7&2&8&9&5&3 \\ 5&9&1&2&3&6&7&8&0&4 \end{matrix}$$

Then the set of diagonal Latin squares, which are isotopic to L (including the first one from the record triple), was constructed. As a result of the second stage it turned out that this diagonal Latin square has 4 diagonal orthogonal mates. Based on these mates on the third stage the record triple with characteristics of 74 was constructed. At the present moment we did not manage to find a better triple in this sense.

3 Proof of Non-existence of Triples of MODLS of Order 10 With Specific Configurations

To search for triples of diagonal Latin squares of order 10 we also applied the SAT approach. According to this approach an initial problem is reduced to a Boolean satisfiability problem (SAT) [7]. In practice, usually, it implies constructing a conjunctive normal form (CNF) followed by checking its satisfiability.

We made a binomial propositional encoding (also called a naive encoding) of the problem of search for a MODLS of order 10. In the corresponding CNF there are 3000 Boolean variables (each cell of each diagonal Latin square in a triple is encoded by 10 Boolean variables) and 1259160 clauses. Then we made 10 SAT instances by adding to the obtained CNF known values of cells from the first 4 rows and 5 first cells of the fifth row of the first diagonal Latin square from a triple. These 10 sets of values were taken from 10 diagonal Latin squares from the first 5 pairs of MODLS of order 10 found in the SAT@home volunteer computing project [6]. Each of constructed sets of 45 cells values corresponds to a family of diagonal Latin squares of order 10 which can be constructed by filling the remaining 55 cells by values that do not violate diagonal Latin square constraints.

So, for each of the constructed 10 SAT instances we considered the problem in the following formulation: for a fixed values of the first 45 cells of the first diagonal Latin square of order 10 to determine if the remaining 55 cells of this square can be filled in such a way that this square participates in a triple of MODLS of order 10. On average, it took 38 minutes to solve a SAT instance from the described set on a computer. In the experiment we used the multithread SAT solver TREENGELING [1]. As a result of the experiment, TREENGELING determined, that all the constructed 10 CNFs are unsatisfiable. Consequently, we have proved that based on 10 described families of diagonal Latin squares it is impossible to construct a triple of MODLS of order 10. Below the first 3 used sets (out of 10) of 45 known values are shown.

0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
1 2 0 4 3 7 9 8 5 6	7 5 1 9 2 8 0 4 6 3	1 2 0 4 3 8 7 9 5 6
7 3 5 9 0 4 8 6 2 1	1 0 3 4 6 7 5 2 9 8	5 6 9 0 7 3 4 8 1 2
3 5 6 8 9 0 4 1 7 2	9 8 4 7 5 2 1 0 3 6	9 8 7 5 6 4 0 1 2 3
4 9 7 2 6	6 7 9 0 8	3 7 5 9 8

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