Roman-type variation of the mixed domination in graphs

J.C. Valenzuela-Tripodoro$^{1,2}$

Department of Mathematics
University of Cadiz
Algeciras, Spain

H. Abdollahzadeh Ahangar$^3$

Department of Basic Sciences
Babol University of Technology
Babol, Iran

T. Haynes $^4$

Department of Mathematics
East Tennesse State University
Johnson City, USA

Abstract
Roman domination in graphs is concerned with the problem of finding a vertex labelling, with minimum weight, satisfying certain conditions. In this work, the authors initiate the study of a generalization to labellings of both vertices and edges in a graph.

Keywords: Roman domination, mixed domination.
1 Introduction

A Roman dominating function is a function $f : V \rightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$. The weight of a Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on $G$ is called the Roman domination number of $G$ and is denoted by $\gamma_R(G)$. Roman domination was introduced by Cockayne et al. [7] in 2004 and was inspired by the work of ReVelle and Rosing [15] and Stewart [16]. Since this introduction, Roman domination and its variations have been studied in the literature, see for example [1,4,5,6,9,13].

We introduce the mixed version of Roman domination as follows. Given a graph $G$, a mixed Roman dominating function (MRDF) of $G$ is a function $f : V \cup E \rightarrow \{0, 1, 2\}$ such that every element, $x \in V \cup E$ for which $f(x) = 0$, is adjacent or incident to at least one element $y \in V \cup E$ with $f(y) = 2$. In other words, we say that an element $x$ for which $f(x) \in \{1, 2\}$ dominates itself, while an element $x$ with $f(x) = 0$ is dominated by $f$ if it is adjacent or incident to at least one element $y$ for which $f(y) = 2$. The minimum weight, $w(f) = \sum_{x \in V \cup E} f(x)$, of a MRDF is the mixed Roman domination number $\gamma^*_R(G)$. A MRDF with minimum weight is called a $\gamma^*_R$-function on $G$. Each MRDF determines a partition of the set $V \cup E = (V_0 \cup E_0) \cup (V_1 \cup E_1) \cup (V_2 \cup E_2)$, where $V_i \cup E_i = \{x \in V \cup E : f(x) = i\}$. For the sake of simplicity, we will denote by $f[x] = f(N_m[x]) = \sum_{v \in N_m[x]} f(v)$, for all $x \in V \cup E$.

For example, consider the complete bipartite graph $K_{r,s}$ with $1 \leq r \leq s$ and partite sets $A$ and $B$ of cardinality $r$ and $s$, respectively. Clearly, assigning a 2 to each vertex in $A$ and 0 to each vertex in $B$ yields a mixed Roman dominating function of cardinality $2r$, so $\gamma^*_R(K_{r,s}) \leq 2r$. We note that since $A$ and $B$ are independent sets, to dominate the edges of $G$, each edge must be assigned a 2 or must be incident to a vertex assigned a 2. Thus, $\gamma^*_R(K_{r,s}) \geq 2r$, and so $\gamma^*_R(K_{r,s}) = 2r$. In particular, for the star $K_{1,n-1}$, we have $\gamma^*_R(K_{1,n-1}) = 2$.

For Roman domination one can think of each vertex representing a city (location) in the Roman Empire and each edge being a road between two cities. A city is secured (protected by the Roman army) if a legion is located

---

1 Research of the author was supported by the Ministry of Education and Science MTM2011-28800-C02-02, Spain. Research of the second co-author was supported in part by the University of Johannesburg.
2 Email: jcarlos.valenzuela@uca.es
3 Email: ha.ahangar@nit.ac.ir
4 Email: haynes@etsu.edu
at that city, and unsecured, otherwise. An unsecured location can be secured by sending a legion from a neighboring secured location to it. In the forth century A.D., Emperor Constantine the Great decreed that a legion could not leave a location to travel to secure another location if doing so left its original location unsecured. Thus, if every location in the Roman Empire was to be protected, any unsecured location must be adjacent to a secured location having at least two legions. In order to minimize costs, at most two legions are placed at any location. Mixed Roman domination is based on the same principles as Roman domination with the additional requirement that “roadways” also be protected from ambush attacks on travelers. In this variation, legions can be placed at a camp on a road as well as stationed in a city, and both cities and roads must be protected. The same guidelines apply to roads (edges) as to cities (vertices). In other words, any unsecured road (edge) must be adjacent to a secured road (edge) with two legions or incident to a secured city (vertex) with two legions. Further, any city or road with two legions can dispatch a legion to secure any unsecured city or road adjacent or incident to it.

2 Basic Properties

We first observe some basic properties of a MRDF.

**Proposition 2.1** For any graph $G$,

$$\gamma^*(G) \leq \gamma^*_R(G) \leq 2\gamma^*(G).$$

Note that the bounds in Proposition 2.1 are sharp. We say that a graph $G$ is a mixed Roman graph if $\gamma^*_R(G) = 2\gamma^*(G)$. Thus, complete bipartite graphs are mixed Roman graphs.

**Proposition 2.2** A graph $G$ is mixed Roman graph if and only if it has a $\gamma^*_R$-function $f = (V_0, V_1, V_2)$ with $|V_1 \cup E_1| = 0$.

**Proposition 2.3** Let $G$ be a graph of order $n \geq 2$. Then $\gamma^*_R(G) = 2$ if and only if $G \in \{K_2, K_{1,n-1}\}$. Further, for $n \geq 3$, $\gamma^*_R(G) = 3$ if and only if $G \in \{K_3, K_{1,n-2} \cup K_1, K_{1,n-1} + e\}$.

3 Main results

Next we give a lower bound on the mixed Roman domination number of a graph in terms of its order, size, and maximum degree.
Proposition 3.1 Let \( G \) be a graph of order \( n \), size \( m \), and maximum degree \( \Delta \geq \delta \geq 1 \). Then
\[
\gamma_R^*(G) \geq \left\lceil \frac{2(m + n)}{2\Delta + 1} \right\rceil.
\]

Corollary 3.2 If \( G \) is an \( r \)-regular graph of order \( n \), then
\[
\gamma_R^*(G) \geq \left\lceil \frac{(r + 2)n}{2r + 1} \right\rceil.
\]

Proposition 3.3 For a non-trivial \( P_n \), \( \gamma_R^*(P_n) = \left\lceil \frac{4n-2}{5} \right\rceil \), if \( n \equiv 0, 1, 2, 3 \pmod{5} \), and \( \left\lceil \frac{4n}{5} \right\rceil + 1 \) otherwise.

Proposition 3.4 For cycles \( C_n \) with \( n \geq 3 \), \( \gamma_R^*(C_n) = \left\lceil \frac{4n}{5} \right\rceil \), if \( n \equiv 0, 2, 3, 4 \pmod{5} \), and \( \left\lceil \frac{4n}{5} \right\rceil + 1 \) otherwise.

We note that the bound given by Corollary 3.2 is also sharp. To illustrate this, we construct a family \( F_k \) of cubic graphs with order \( 7k \) for any even integer \( k \geq 2 \) as follows: Let \( H_k \) be the union of \( k \) claws \( K_{1,3} \) where each claw has center \( v_i \) for \( 1 \leq i \leq k \), and let \( M_k \) be the union of \( 3k/2 \) edges. Construct a graph \( G \) from \( H_k \cup M_k \) by adding \( 6k \) new edges, each joining a vertex in \( H_k \) to a vertex in \( M_k \), in such a way that the resulting graph is cubic. Note that each of the additional \( 6k \) edges is dominated by the edges of \( M_k \). Thus, the set \( S = E(M_k) \cup \{v_i | 1 \leq i \leq k\} \) is a mixed Roman dominating set of \( G \), and assigning a 2 to each element of \( S \) and a 0 to all other elements of \( G \) yields a mixed Roman dominating function with weight \( 2|S| = 2(3k/2 + k) = 5k = 5(7k)/7 = 5n/7 \). For an example where \( k = 2 \), see Figure 1.

Fig. 1. \( G \in F_2 \)

Proposition 3.5 For any graph \( G \) of order \( n \), \( \gamma_R^k(G) \leq n \).

Proposition 3.6 If \( K_n \) is the complete graph on \( n \) vertices, then \( \gamma_R^k(K_n) = n \).
Next, we give a sharp upper bound for the mixed Roman domination number of a connected graph and characterize the graphs attaining this bound. To aid in the characterization, we let $G(a, b, c)$ denote the graph obtained from a non-trivial star $K_{1,n-1}$ with center $v$ by adding edges from its complement such that $G(a, b, c) - v = aK_1 \cup bK_2 \cup cP_3$, and for $j \leq a$, we let $G_j(a, b, c)$ be the graph obtained from $G(a, b, c)$ by subdividing (once) $j$ pendant edges. (See Figure 2). Let $\mathcal{H}$ be the family of graphs $\mathcal{H} = \{G(a, b, c), G_j(a, b, c) : a, b, c \geq 0, j \leq a\}$ satisfying that if $G \in \mathcal{H}$ and $(b, c) \in \{(0, 0), (1, 0)\}$, then either $G = G(0, 1, 0) = K_3$ or $a > j$.

![Fig. 2. $G(4, 1, 1)$ and $G_3(4, 2, 0)$](image)

**Proposition 3.7** Let $G$ be a connected graph of order $n \geq 2$, size $m$, and $\Delta(G) \geq 1$. Then $\gamma_{R}^*(G) \leq m + n - 2\Delta(G) + 1$, with equality if and only if $G \in \mathcal{H}$.

**References**


