Long cycles in 3-polytopes

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Let $G$ be the skeleton of a 3-polytope. By Steinitz’s theorem, we may consider equivalently 3-connected planar graphs.

**Problem.** Does there exist $\alpha > 0$ such that almost all 3-connected planar graphs with $n$ vertices have a cycle of length $\alpha n$?

Clearly we may consider instead graphs with $n$ edges, or with $n$ faces. It is known that there are 3-connected planar graphs whose longest cycle has length at most $9n^{\log_3 2}$; in fact there is always a cycle of length $\Omega(n^{\log_3 2})$ (see [1] and the references therein).

For triangulations (which correspond to simplicial polytopes, that is, every face is a triangle) we have a positive answer to the problem above. A triangulation $T$ is 4-connected if it has no separating triangle, that is, a triangle which is not a face. After removing the interior of all separating triangles we get the 4-connected core $C$ of the triangulation, which is Hamiltonian by Tutte’s theorem. Note that a Hamiltonian cycle of $C$ gives rise to a cycle of $T$ of the same length.

It has been shown [2] that with high probability the 4-connected core of a triangulation with $n$ edges has $cn$ edges for some constant $c > 0$. Hence this implies the existence of a cycle of linear length in $T$.

Using recent results on planar graphs [3], an positive solution to the problem above would imply that a random planar graph with $n$ vertices has almost surely a cycle of linear length.

**References**

